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MINIMUM PRINCIPLES IN THE LINEAR THEORY OF MICROPOLAR PIEZOELECTRICITY

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Abstract. In this paper we will approach the linear theory of micropolar piezoelectricity with polarization gradient in the case of quasielectrostatic approximation. Using the reciprocal relation, we derive minimum principles in the considered theory.

1 Introduction

The theory of piezoelectricity in the case of quasielectrostatic fields has been considered in various papers (see, e.g. [4], [5], [7]). Depending on the degree of finesse imposed by a practical problem, there are various mathematic models of piezoelectric materials to be considered. Thus in [2], [3], [6] the piezoelectric model with polarization gradient has been introduced. In [1], we presented the linear theory of a micropolar piezolectric model with polarization gradient. This theory was established using a variational principle of the Hamilton type. By using the method suggested by D. Ieşan [4], we proved the reciprocity theorem and the uniqueness theorem. Reiss and Haug [8], have established a general minimum principle for a wide class of linear initial and mixed boundary–value problems. For the linear theory of piezoelectricity, the minimum principle was established by D. Ieşan [4].

In this paper we extend these results to the quasistatic theory of micropolar piezoelectricity with polarization gradient. Assuming a solution does exist, we seek a functional that attains its minimum at this solution. All considerations are presented here in a tensorial form.

2 Quasistatic electric fields

We consider a body that at the time $t = 0$ occupies the region D of the three dimensional Euclidean space and is bounded by the piecewise smooth surface S. The motion of the body is referred to a fixed system of rectangular Cartesian axes Ox_i ($i = 1, 2, 3$). Letters in boldface stand for tensors of an order $p \geq 1$. In [1] we derived a linear theory for micropolar piezoelectric materials where the polarization gradient is added to the set of the independent constitutive variables. The local field equations and the associated boundary conditions which govern the motion were obtained by using the generalized formulation of the Hamilton's principle. The fundamental system of field equations consists in the following:

– the equations of motions and the equations of the quasistatic electric field

$$\begin{aligned} &\nabla \cdot \boldsymbol{\tau}^{\top} + \boldsymbol{F} = \rho \ddot{\boldsymbol{u}}, \\ &\nabla \cdot \boldsymbol{\mu}^{\top} + \boldsymbol{R}[\boldsymbol{\tau}] + \boldsymbol{G} = \boldsymbol{I}\ddot{\boldsymbol{w}}, \\ &\nabla \cdot \boldsymbol{\pi}_1^{\top} - \nabla\varphi + \boldsymbol{\pi} + \boldsymbol{E}^0 = 0, \\ &\nabla \cdot \boldsymbol{P} - \varepsilon_0 \Delta\varphi = f, \quad \text{in } D; \end{aligned} \tag{2.1}$$

– the constitutive equations

$$\begin{aligned} &\boldsymbol{\tau} = \boldsymbol{A}[\boldsymbol{e}] + \boldsymbol{A}_1[\boldsymbol{\kappa}] - \boldsymbol{A}_2\boldsymbol{P} - \boldsymbol{B}_2[(\nabla \cdot \boldsymbol{P})^{\top}], \\ &\boldsymbol{\mu} = \boldsymbol{e}\boldsymbol{A}_1 + \boldsymbol{B}[\boldsymbol{\kappa}] - \boldsymbol{B}_1\boldsymbol{P} - \boldsymbol{B}_3[(\nabla \cdot \boldsymbol{P})^{\top}], \\ &\boldsymbol{\pi} = \boldsymbol{e}\boldsymbol{A}_2 + \boldsymbol{\kappa}\boldsymbol{B}_1 - \boldsymbol{A}_3\boldsymbol{P} - \boldsymbol{A}_4[(\nabla \cdot \boldsymbol{P})^{\top}], \\ &\boldsymbol{\pi} = -\boldsymbol{e}\boldsymbol{B}_2 - \boldsymbol{\kappa}\boldsymbol{B}_3 + \boldsymbol{P}\boldsymbol{A}_4 + \boldsymbol{D}[(\nabla \cdot \boldsymbol{P})^{\top}] + \boldsymbol{C}; \end{aligned} \tag{2.2}$$

and the relations

$$\boldsymbol{e} = (\nabla \cdot \boldsymbol{u})^{\top} + \boldsymbol{R}\boldsymbol{w}, \ \boldsymbol{\kappa} = (\nabla \cdot \boldsymbol{w})^{\top}, \ \boldsymbol{E} = -\nabla\varphi \tag{2.3}$$

In the above relations we have used the following notations:

– $\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{P}, \varphi$ are the displacement, the microrotation, the polarization, the electric potential;
– $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{E}^0, f$ are the body force, the body–couple, the electric field, the volume density of free charge;
– $\boldsymbol{R}, \boldsymbol{I}, \rho, \varepsilon_0$ are the alternating tensor, the microinertia tensor, the mass density, the permitivity of the vacuum;
– $\boldsymbol{\tau}, \boldsymbol{\mu}, \boldsymbol{\pi}, \boldsymbol{\pi}_1$ are the stress tensor, the couple–stress tensor, the local electric force, the dipolar electric tensor;
– $\boldsymbol{A}, \boldsymbol{A}_i$ $(i = 1, 2, 3, 4)$, $\boldsymbol{B}, \boldsymbol{B}_j$ $(j = 1, 2, 3)$ and $\boldsymbol{C}$ are the constitutive tensors;
– $\boldsymbol{e}, \boldsymbol{\kappa}$ are the deformation tensors and $\boldsymbol{E}$ is the Maxwell's electric field.

We assume that the constitutive tensors satisfy the relations:

$$\begin{aligned} &\boldsymbol{I} = \rho\boldsymbol{J}, \ \boldsymbol{J}\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{J}\boldsymbol{b}, \ \boldsymbol{a} \cdot \boldsymbol{J}\boldsymbol{a} > 0, \\ &\boldsymbol{A}_3\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{A}_3\boldsymbol{b}, \ \boldsymbol{a} \cdot \boldsymbol{A}[\boldsymbol{b}] = \boldsymbol{b} \cdot \boldsymbol{A}[\boldsymbol{a}], \\ &\boldsymbol{a} \cdot \boldsymbol{B}[\boldsymbol{b}] = \boldsymbol{b} \cdot \boldsymbol{B}[\boldsymbol{a}], \ \boldsymbol{a} \cdot \boldsymbol{D}[\boldsymbol{b}] = \boldsymbol{b} \cdot \boldsymbol{D}[\boldsymbol{a}]. \end{aligned} \tag{2.5}$$

We consider the following boundary conditions:

$$\begin{aligned} &\boldsymbol{u} = \boldsymbol{u}' \text{ on } S_1 \times I, \ \ \boldsymbol{t} = \boldsymbol{\tau}^{\top}\boldsymbol{n} = \boldsymbol{t}' \text{ on } S_2 \times I, \\ &\boldsymbol{w} = \boldsymbol{w}' \text{ on } S_3 \times I, \ \boldsymbol{m} = \boldsymbol{\mu}^{\top}\boldsymbol{n} = \boldsymbol{m}' \text{ on } S_4 \times I, \\ &\varphi = \varphi' \text{ on } S_5 \times I, \ \ (\varepsilon_0 < \nabla\varphi > -\boldsymbol{P}) \cdot \boldsymbol{n} = f' \text{ on } S_6 \times I, \\ &\boldsymbol{P} = \boldsymbol{P}' \text{ on } S_7 \times I, \ \boldsymbol{\pi}_1^{\top}\boldsymbol{n} = \boldsymbol{s}' \text{ on } S_8 \times I, \end{aligned} \tag{2.6}$$

where S_i $(i = 1, 2, ..., 8)$ denote subsets of S so that

$$\begin{aligned}
&S_1 \cup \overline{S}_2 = S_3 \cup \overline{S}_4 = S_5 \cup \overline{S}_6 = S_7 \cup \overline{S}_8 = S,\\
&S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = \emptyset,\ I = [0, t_1),
\end{aligned}$$

t_1 is an instant that may be infinite, and $\mathcal{F}' = \{\boldsymbol{u}', \boldsymbol{t}', \boldsymbol{w}', \boldsymbol{m}', \varphi', f', \boldsymbol{P}', \boldsymbol{s}'\}$ are prescribed functions.

We adjoin the initial conditions

$$\begin{aligned}
&\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{a}'(\boldsymbol{x}),\ \dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{b}'(\boldsymbol{x}),\\
&\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{c}'(\boldsymbol{x}),\ \dot{\boldsymbol{w}}(\boldsymbol{x}, 0) = \boldsymbol{d}'(\boldsymbol{x}).
\end{aligned} \tag{2.7}$$

In the liner theory the energy density of deformation and polarization is taken to be

$$W = W_1 + C[(\nabla \cdot \boldsymbol{P})^\top],$$

where

$$\begin{aligned}
W_1 = {}&\frac{1}{2}\boldsymbol{e} \cdot \boldsymbol{A}[\boldsymbol{e}] + \boldsymbol{e} \cdot \boldsymbol{A}_1[\boldsymbol{\kappa}] - \boldsymbol{e} \cdot \boldsymbol{A}_2\boldsymbol{P} - \boldsymbol{e} \cdot \boldsymbol{B}_2[(\nabla \cdot \boldsymbol{P})^\top] +\\
&+ \frac{1}{2}\boldsymbol{\kappa} \cdot \boldsymbol{B}[\boldsymbol{\kappa}] - \boldsymbol{\kappa} \cdot \boldsymbol{B}_1\boldsymbol{P} - \boldsymbol{\kappa} \cdot \boldsymbol{B}_3[(\nabla \cdot \boldsymbol{P})^\top] +\\
&+ \frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{A}_3\boldsymbol{P} + \boldsymbol{P} \cdot \boldsymbol{A}_4[(\nabla \cdot \boldsymbol{P})^\top] + \frac{1}{2}(\nabla \cdot \boldsymbol{P})^\top \cdot \boldsymbol{D}[(\nabla \cdot \boldsymbol{P})^\top].
\end{aligned} \tag{2.8}$$

In [1] we proved the reciprocal relation for the mixed problem (2.1), (2.6), (2.7). If the boundary conditions (2.6) are homogeneous, then the reciprocal relation has the following form:

$$\begin{aligned}
&\int_D [\boldsymbol{f}^{(1)} * \boldsymbol{u}^{(2)} + \boldsymbol{g}^{(1)} * \boldsymbol{w}^{(2)} + \gamma * (\boldsymbol{E}^{0(1)} * \boldsymbol{P}^{(2)} - f^{(1)} * \varphi^{(2)})]dv =\\
&= \int_D [\boldsymbol{f}^{(2)} * \boldsymbol{u}^{(1)} + \boldsymbol{g}^{(2)} * \boldsymbol{w}^{(1)} + \gamma * (\boldsymbol{E}^{0(2)} * \boldsymbol{P}^{(1)} - f^{(2)} * \varphi^{(1)})]dv,
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
&\gamma(t) = t,\ \boldsymbol{f}^{(\alpha)} = \gamma * \boldsymbol{F}^{(\alpha)} + \rho(t\boldsymbol{b}'^{(\alpha)} + \boldsymbol{a}'^{(\alpha)}),\\
&\boldsymbol{g}^{(\alpha)} = \gamma * \boldsymbol{G}^{(\alpha)} + \boldsymbol{I} \cdot (t\boldsymbol{d}'^{(\alpha)} + \boldsymbol{c}'^{(\alpha)}),\ \alpha = 1, 2,
\end{aligned} \tag{2.10}$$

$$(\boldsymbol{a} * \boldsymbol{b})(\boldsymbol{x}, t) = (a_i * b_i)(\boldsymbol{x}, t) = \int_0^t a_i(\boldsymbol{x}, t - \tau)b_i(\boldsymbol{x}, \tau)d\tau. \tag{2.11}$$

In what follows, we write the initial and mixed boundary–value problem in an alternative form in which the initial conditions are incorporated in the field equations.

Theorem 2.1. *The functions* $\mathcal{F} = \{\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{P}, \varphi, \boldsymbol{\tau}, \boldsymbol{\mu}, \boldsymbol{\pi}, \boldsymbol{\pi}_1\}$ *satisfy Eqs.* (2.1) *and the initial conditions* (2.7) *if and only if*

$$\begin{aligned}
&\gamma * \nabla \cdot \boldsymbol{\tau}^\top + \boldsymbol{f} = \rho\boldsymbol{u},\ \gamma * (\nabla \cdot \boldsymbol{\mu}^\top + \boldsymbol{R}[\boldsymbol{\tau}]) + \boldsymbol{g} = \boldsymbol{I}\boldsymbol{w},\\
&\gamma * (\nabla \cdot \boldsymbol{\pi}_1^\top - \nabla\varphi + \boldsymbol{\pi} + \boldsymbol{E}^0) = 0,\ \gamma * (\nabla \cdot \boldsymbol{P} - \varepsilon_0 \Delta\varphi) = \gamma * f,
\end{aligned} \tag{2.12}$$

where

$$\gamma(t) = t,\ \boldsymbol{f} = \gamma * \boldsymbol{F} + \rho(t\boldsymbol{b}' + \boldsymbol{a}'),\ \boldsymbol{g} = \gamma * \boldsymbol{G} + \boldsymbol{I}(t\boldsymbol{d}' + \boldsymbol{c}'). \tag{2.13}$$

PROOF. We assume that the $\mathcal{F}$ functions satisfy (2.1) and (2.7). Taking the convolution of the relation $(2.1)_1$ with $\gamma(t)$, we get

$$\gamma * \nabla \cdot \boldsymbol{\tau}^{\top} + \gamma * \boldsymbol{F} = \rho \boldsymbol{u} - \rho[t\dot{\boldsymbol{u}}(\boldsymbol{x},0) + \boldsymbol{u}(\boldsymbol{x},0)]. \tag{2.14}$$

By using the notations (2.7) and (2.13), from (2.14) we obtain the first (2.12) relation. Similarly, we prove the other (2.12) relations.

Now, we assume that the functions $\mathcal{F}$ satisfy $(2.12)_1$ and (2.13). From this relations and the properties of the convolution, we have

$$\gamma * (\nabla \cdot \boldsymbol{\tau}^{\top} + \boldsymbol{F}) + \rho(t\boldsymbol{b}' + \boldsymbol{a}') = \rho[\gamma * \ddot{\mathrm{u}} + t\dot{\boldsymbol{u}}(\boldsymbol{x},0) + \boldsymbol{u}(\boldsymbol{x},0)]. \tag{2.15}$$

Deriving (2.15) in relation with the time, we obtain

$$\int_0^t (\nabla \cdot \boldsymbol{\tau}^{\top} + \boldsymbol{F})d\tau + \rho \boldsymbol{b}' = \rho \int_0^t \ddot{\mathrm{u}}(x,\tau)d\tau + \rho\dot{\boldsymbol{u}}(\boldsymbol{x},0) \tag{2.16}$$

because

$$\frac{d}{dt}(\gamma * v) = \int_0^t v(\boldsymbol{x},\tau)d\tau.$$

For $t=0$, from (2.15) and (2.16), it result (2.7). With the help of the initial conditions and the properties of the convolution, from (2.15) it results $(2.1)_1$. Analogously the other (2.1) relations are demonstrated. □

3 Minimum principles

We consider the (2.12) problem with the (2.6) boundary conditions in the case of

$$\mathcal{F}' = 0. \tag{3.1}$$

Let $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{P}, \varphi) = (U_i)_{i=1,2,\dots,10}$. The $\boldsymbol{U}$ vector is named *admissible vector* if

(i) $U_i \in C^2[D \times [0,\infty)],\ U_i \in C^{1,0}[\overline{D} \times [0,\infty)]$,
(ii) The functions $\mathcal{F}_1 = (U_i, \dot{U}_i, U_{i,j}, U_{i,jk}),\ j,k = 1,2,3$ are bounded at infinity, that is $\lim\limits_{t\to\infty} \mathcal{F}_1(\boldsymbol{x},t)$ exists for each $\boldsymbol{x} \in \overline{D}$.

Let X be the set of all functions which satisfy (i) and (ii) conditions.

By a *solution of the problem* we mean an admissible ten–vector field $\boldsymbol{U}$ that satisfies the Eqs. (2.12) and the homogeneous boundary conditions.

Let $\mathcal{U}$ be the set of all admissible vectors which satisfy (3.1).

We introduce the operators M_i $(i = 1,2,\dots,10)$, on $\mathcal{U}$, defined by

$$\begin{aligned} M_i\boldsymbol{U} &= \rho u_i - \gamma * \tau_{ji,j}, \\ M_{i+3}\boldsymbol{U} &= I_{ij}\varphi_j - \gamma * (\mu_{ji,j} + \varepsilon_{ijk}\tau_{jk}), \\ M_{i+6}\boldsymbol{U} &= \gamma * (\varphi_{,i} - \pi_i - \pi_{ji,j}), \\ M_{10}\boldsymbol{U} &= \gamma * (P_{i,i} - \varepsilon_0\varphi_{,ii}),\ i,j,k = 1,2,3. \end{aligned} \tag{3.2}$$

If we introduce the notations

$$\begin{aligned} &\boldsymbol{MU} = (M_1\boldsymbol{U}, M_2\boldsymbol{U}, ..., M_{10}\boldsymbol{U}),\\ &\boldsymbol{L} = (\boldsymbol{f}, \boldsymbol{g}, \gamma * \boldsymbol{E}^0, \gamma * f) = (L_i)_{i=1,2,...,10} \end{aligned} \tag{3.3}$$

then the equations (2.13) can be written in the form of one vector equation

$$\boldsymbol{MU} = \boldsymbol{L}. \tag{3.4}$$

Let Y be the set of all continuous functions on $\overline{D} \times [0, \infty)$ that are bounded at infinity.

We consider $g(t), h(t)$ two functions defined on $[0, \infty)$, and we assume that h is continuous and positive on $[0, \infty)$, and has a finite limit at infinity. The function g is named *admissible weight function* if g satisfy the following conditions:

(a) $\int_0^\infty \int_0^\infty g^{(k)}(t+s)ds\,dt$ exists for $k \in \mathbb{N}$, where

$$g^{(k)} = \frac{\partial^k g}{\partial t^k}; \tag{3.5}$$

(b) $g(t) = \int_0^\infty h(s)e^{-st}ds, \; t \in [0, \infty)$.

Let Γ be the set of admissible weight functions.

We introduce the notation

$$[u, v]_g = \int_0^\infty \int_0^\infty \int_D g(t+s)u(\boldsymbol{x}, t)v(\boldsymbol{x}, s)dt\,ds\,dv \tag{3.6}$$

for $u, v \in Y, \; g \in \Gamma$.

We note that

$$[u, v]_g = [v, u]_g. \tag{3.7}$$

In what follows we denote by Ψ^* the Laplace transform with respect to time of the function Ψ, defined by

$$\Psi^*(\boldsymbol{x}, q) = \int_0^\infty \Psi(\boldsymbol{x}, t)e^{-qt}dt, \; q \in \mathbb{R}. \tag{3.8}$$

From (3.5) and (3.6) we have

$$[u, v]_g = \int_D \int_0^\infty h(s)u^*(\boldsymbol{x}, s)v^*(\boldsymbol{x}, s)ds\,dv \tag{3.9}$$

for any $u, v \in X$.

We introduce the notation

$$[\boldsymbol{U}, \boldsymbol{U}']_g = \sum_{i=1}^{10} [U_i, U_i']_g, \tag{3.10}$$

for $\boldsymbol{U}, \boldsymbol{U}' \in \mathcal{U}, \; g \in \Gamma$.

We consider two external data systems $\boldsymbol{L}^{(\alpha)}$, $\alpha = 1, 2$, with null boundary data, and we denote by $\boldsymbol{U}^{(\alpha)}$, $\alpha = 1, 2$, the solutions of the problem (3.4). We have

$$\boldsymbol{MU}^{(\alpha)} = \boldsymbol{L}^{(\alpha)}, \; \alpha = 1, 2. \tag{3.11}$$

Theorem 3.1. *Assume that*

1) *the density field ρ is strictly positive;*
2) *the constitutive coefficients satisfy the conditions (2,4), (2.5);*
3) *the energy density of deformation and polarization W_1 is a positive semi-definite quadratic form.*

Then the operator $\boldsymbol{M}$ satisfy the following conditions:

$$
\begin{aligned}
&[\boldsymbol{M}\boldsymbol{U},\boldsymbol{U}]_g = [\boldsymbol{U},\boldsymbol{M}\boldsymbol{U}]_g,\\
&[\boldsymbol{M}\boldsymbol{U},\boldsymbol{U}]_g \geq 0,
\end{aligned}
\tag{3.12}
$$

for any $\boldsymbol{U},\boldsymbol{U}' \in \mathcal{U}$.

Proof. We say that an operator M is g-positive if (3.12) holds.

The reciprocal relation (2.9) can be written in the form:

$$\int_D \boldsymbol{L}^{(1)} * \boldsymbol{U}^{(2)} dv = \int_D \boldsymbol{L}^{(2)} * \boldsymbol{U}^{(1)} dv. \tag{3.13}$$

If we take the Laplace transform of this relation we obtain, with the help of Borel's theorem

$$(u * v)^* = u^* v^*, \tag{3.14}$$

the relation

$$\int_D \boldsymbol{L}^{(1)^*} \boldsymbol{U}^{(2)^*} dv = \int_D \boldsymbol{L}^{(2)^*} \boldsymbol{U}^{(1)^*} dv. \tag{3.15}$$

Clearly, from (3.15), we get

$$\int_0^\infty \int_D h(r) \boldsymbol{L}^{(1)^*} \boldsymbol{U}^{(2)^*} dv\, dr = \int_0^\infty \int_D h(r) \boldsymbol{L}^{(2)^*} \boldsymbol{U}^{(1)^*} dv\, dr. \tag{3.16}$$

By using (3.7) and (3.9), from (3.16) we obtain

$$[\boldsymbol{M}\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}]_g = [\boldsymbol{M}\boldsymbol{U}^{(2)}, \boldsymbol{U}^{(1)}]_g = [\boldsymbol{U}^{(1)}, \boldsymbol{M}\boldsymbol{U}^{(2)}]_g, \tag{3.17}$$

that is the operator $\boldsymbol{M}$ is a g-symmetric operator.

In [1] we proved the following relation:

$$
\begin{aligned}
&\int_D [\boldsymbol{F}^{(1)}(r) \cdot \boldsymbol{u}^{(2)}(s) + \boldsymbol{G}^{(1)}(r) \cdot \boldsymbol{w}^{(2)}(s) + \boldsymbol{E}^{0^{(1)}}(r) \cdot \boldsymbol{P}^{(2)}(s) + \varphi^{(1)}(r) \cdot f^{(2)}(s)] dv - \\
&\qquad - \int_D [\rho \ddot{\boldsymbol{u}}^{(1)}(r) \cdot \boldsymbol{u}^{(2)}(s) + \ddot{\boldsymbol{w}}^{(1)}(r) \cdot \boldsymbol{I} \boldsymbol{w}^{(2)}(s)] dv = \\
&\qquad = \int_D \{\boldsymbol{\tau}^{(1)}(r) \cdot \boldsymbol{e}^{(2)}(s) + \boldsymbol{\mu}^{(1)}(r) \cdot \boldsymbol{\kappa}^{(2)}(s) - \boldsymbol{\pi}^{(1)}(r) \cdot \boldsymbol{P}^{(2)}(s) + \\
&\qquad + \varepsilon_0 \boldsymbol{E}^{(1)}(r) \cdot \boldsymbol{E}^{(2)}(s) + [\pi_1^{(1)}(r) - \boldsymbol{\xi}^{(1)}(r)] (\nabla \cdot \boldsymbol{P}^{(2)}(s))^\top\} dv,
\end{aligned}
\tag{3.18}
$$

for any $r, s \in [0, t_1)$.

In the relation (3.18), we denoted $\boldsymbol{\xi} = \boldsymbol{C}\cdot\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is Kronecker's tensor.

We now consider in (3.18) $(\boldsymbol{F}^{(\alpha)}, \boldsymbol{G}^{(\alpha)}, \boldsymbol{E}^{0(\alpha)}, f^{(\alpha)}) = (\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{E}^0, f)$, $U^{(\alpha)} = U$, $\alpha = 1, 2$, $\boldsymbol{U} \in \mathcal{U}$, $r = \tau$, $s = t - \tau$ and integrate with respect to τ from 0 to t. We obtain

$$\begin{aligned}&\int_D (\boldsymbol{F} * \boldsymbol{u} + \boldsymbol{G} * \boldsymbol{w} + \boldsymbol{E}^0 * \boldsymbol{P} + f * \varphi) dv - \\ &- \int_D (\rho \ddot{\boldsymbol{u}} * \boldsymbol{u} + \ddot{\boldsymbol{w}} * \boldsymbol{I}\boldsymbol{w}) dv = \int_D [\boldsymbol{\tau} * \boldsymbol{e} + \boldsymbol{\mu} * \boldsymbol{\kappa} - \boldsymbol{\pi} * \boldsymbol{P} + \\ &+ \varepsilon_0 \boldsymbol{E} * \boldsymbol{E} + (\boldsymbol{\pi}_1 - \boldsymbol{\xi}) * (\nabla \cdot \boldsymbol{P})^{\top}] dv.\end{aligned} \tag{3.19}$$

We take in (3.19) the convolution with $\gamma(t) = t$, and with the aid of the relations

$$(\gamma * \ddot{\boldsymbol{u}})(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t) - t\dot{\boldsymbol{u}}(\boldsymbol{x}, 0) - \boldsymbol{u}(\boldsymbol{x}, 0), \tag{3.20}$$

we obtain

$$\begin{aligned}&\int_D \gamma * [\boldsymbol{\tau} * \boldsymbol{e} + \boldsymbol{\mu} * \boldsymbol{\kappa} - \boldsymbol{\pi} * \boldsymbol{P} + \varepsilon_0 \boldsymbol{E} * \boldsymbol{E} + (\boldsymbol{\pi}_1 - \boldsymbol{\xi}) * (\nabla \cdot \boldsymbol{P})^{\top}] dv = \\ &= \int_D [\boldsymbol{f} * \boldsymbol{u} + \boldsymbol{g} * \boldsymbol{w} + (\gamma * \boldsymbol{E}^0) * \boldsymbol{P} + (\gamma * f) * \varphi] dv - \\ &- \int_D (\rho \boldsymbol{u} * \boldsymbol{u} + \boldsymbol{w} * \boldsymbol{I}\boldsymbol{w}) dv.\end{aligned} \tag{3.21}$$

We take the Laplace transform of this relation. We obtain, with the aid of (3.14) that

$$\begin{aligned}&\int_D [\boldsymbol{f}^* \cdot \boldsymbol{u}^* + \boldsymbol{g}^* \cdot \boldsymbol{w}^* + (\gamma * \boldsymbol{E}^0)^* \cdot \boldsymbol{P}^* + (\gamma * f)^* \varphi^*] dv - \\ &- \int_D (\rho \boldsymbol{u}^* \cdot \boldsymbol{u}^* + \boldsymbol{w}^* \cdot \boldsymbol{I}\boldsymbol{w}^*) dv = \\ &= \int_D \gamma^* [\boldsymbol{\tau}^* \cdot \boldsymbol{e}^* + \boldsymbol{\mu}^* \cdot \boldsymbol{\kappa}^* - \boldsymbol{\pi}^* \cdot \boldsymbol{P}^* + \varepsilon_0 \boldsymbol{E}^* \cdot \boldsymbol{E}^* + (\boldsymbol{\pi}_1^* - \boldsymbol{\xi}^*) \cdot (\nabla \cdot \boldsymbol{P}^*)^{\top}] dv.\end{aligned} \tag{3.22}$$

We multiply in (3.22) with $h(s)$ and integrate with respect to s from 0 to ∞, we obtain, with the aid of (3.9), that

$$\begin{aligned}[\boldsymbol{M}\boldsymbol{U}, \boldsymbol{u}]_g = &\int_0^\infty \int_D h(s) [\rho \boldsymbol{u}^*(s) \cdot \boldsymbol{u}^*(s) + \boldsymbol{w}^*(s) \cdot \boldsymbol{I}\boldsymbol{w}^*(s)] dv\, ds + \\ &+ \int_0^\infty \int_D 2h(s) \gamma^*(s) W_1^*(s) dv\, ds + \\ &+ \int_0^\infty \int_D \varepsilon_0 h(s) \gamma^*(s) \boldsymbol{E}^*(s) \boldsymbol{E}^*(s) dv\, ds.\end{aligned} \tag{3.23}$$

For $\gamma(t) = t$, $\zeta(t) = 1$, we have

$$\gamma^*(s) = s^{-2},\ \zeta^*(s) = s^{-1},\ (\zeta * u)^*(s) = \zeta^*(s) u^*(s) = s^{-1} u^*(s), \tag{3.24}$$

for every $u \in Y$.

If we denote

$$\widetilde{u}(\boldsymbol{x},t)=(\zeta * u)(\boldsymbol{x},t)=\int_0^t u(\boldsymbol{x},\tau)d\tau, \tag{3.25}$$

then we get

$$s^{-1}u^{*}=\widetilde{u}^{*}. \tag{3.26}$$

With the help of the relations (2.8), (3.6), (3.9), (3.24)–(3.26), the relation (3.23) can be written in the following form:

$$\begin{aligned}
[\boldsymbol{MU},\boldsymbol{U}]_g = & \int_D\int_0^\infty\int_0^\infty g(t+s)[\rho \boldsymbol{u}(t)\cdot\boldsymbol{u}(s)+\boldsymbol{w}(t)\cdot\boldsymbol{I}\boldsymbol{w}(s)]dv\,ds\,dt+\\
& +\int_D\int_0^\infty\int_0^\infty 2g(t+s)\{\frac{1}{2}\widetilde{\boldsymbol{e}}(t)\cdot\boldsymbol{A}[\widetilde{\boldsymbol{e}}(s)]+\widetilde{\boldsymbol{e}}(t)\cdot\boldsymbol{A}_1[\widetilde{\boldsymbol{\kappa}}(s)]-\widetilde{\boldsymbol{e}}(t)\cdot\boldsymbol{A}_2\widetilde{\boldsymbol{P}}(s)-\\
& -\widetilde{\boldsymbol{e}}(t)\cdot\boldsymbol{B}_2[(\nabla\cdot\widetilde{\boldsymbol{P}}(s))^{\top}]+\frac{1}{2}\widetilde{\boldsymbol{\kappa}}(t)\cdot\boldsymbol{B}[\widetilde{\boldsymbol{\kappa}}(s)]-\widetilde{\boldsymbol{\kappa}}(t)\cdot\boldsymbol{B}_1\widetilde{\boldsymbol{P}}(s)-\\
& -\widetilde{\boldsymbol{\kappa}}(t)\cdot\boldsymbol{B}_3[(\nabla\cdot\widetilde{\boldsymbol{P}}(s))^{\top}]+\frac{1}{2}\widetilde{\boldsymbol{P}}(t)\cdot\boldsymbol{A}_3\widetilde{\boldsymbol{P}}(s)+\\
& +\widetilde{\boldsymbol{P}}(t)\cdot\boldsymbol{A}_4[(\nabla\cdot\widetilde{\boldsymbol{P}}(s))^{\top}]+\frac{1}{2}(\nabla\cdot\widetilde{\boldsymbol{P}}(t))^{\top}\cdot\boldsymbol{D}(\nabla\cdot\widetilde{\boldsymbol{P}}(s))^{\top}]\}dv\,ds\,dt+\\
& +\int_D\int_0^\infty\int_0^\infty \varepsilon_0 g(t+s)\widetilde{\boldsymbol{E}}(t)\cdot\widetilde{\boldsymbol{E}}(s)dv\,ds\,dt.
\end{aligned} \tag{3.27}$$

By the hypotheses 1), 2), 3) we obtain that

$$[\boldsymbol{MU},\boldsymbol{U}]_g\geq 0, \tag{3.28}$$

that is the operator M is g–positive. □

We define the following functional

$$\Phi_g(U)=[\boldsymbol{MU},U]_g-2[\boldsymbol{U},\boldsymbol{L}]_g,\ \boldsymbol{U}\in\mathcal{U},\ g\in\Gamma. \tag{3.29}$$

Theorem 3.2 (Minimum principles). *Assume that*

1°. *the density field ρ is strictly positive;*
2°. *the constitutive coefficients satisfy the conditions (2.4), (2.5);*
3°. *the energy density of deformation and polarization W_1 is a semi–definite quadratic form.*

Further, let $\boldsymbol{U}\in\mathcal{U}$ be a solution of the boundary–initial–value problem (2.1)–(2.3), (2.7), (3.1). Then

$$\Phi_g(\boldsymbol{U})\leq\Phi_g(\boldsymbol{U}') \tag{3.30}$$

for every $\boldsymbol{U}'\in\mathcal{U}$.

PROOF. By the hypotheses 1°, 2°, 3° and theorem 3.1 we have that $\boldsymbol{M}$ is a g–positive operator. Let $\boldsymbol{U}$ be a solution of the considered problem, $\boldsymbol{U}'\in\mathcal{U}$, and $\boldsymbol{U}_0=\boldsymbol{U}'-\boldsymbol{U}$.

We have

$$\begin{aligned}\Phi_g(\boldsymbol{U}') &= \phi_g(\boldsymbol{U}+\boldsymbol{U}_0) = [\boldsymbol{M}(\boldsymbol{U}+\boldsymbol{U}_0), \boldsymbol{U}+\boldsymbol{U}_0]_g - 2[\boldsymbol{U}+\boldsymbol{U}_0, \boldsymbol{L}]_g = \\ &= [\boldsymbol{M}\boldsymbol{U}, \boldsymbol{U}]_g - 2[\boldsymbol{U}, \boldsymbol{L}]_g + 2[\boldsymbol{M}\boldsymbol{U} - \boldsymbol{L}, \boldsymbol{U}_0]_g + [\boldsymbol{M}\boldsymbol{U}_0, \boldsymbol{U}_0] = \\ &= \Phi_g(\boldsymbol{U}) + 2[\boldsymbol{M}\boldsymbol{U} - \boldsymbol{L}, \boldsymbol{U}_0]_g + [\boldsymbol{M}\boldsymbol{U}_0, \boldsymbol{U}_0]_g.\end{aligned} \tag{3.31}$$

Because $\boldsymbol{U}$ is a solution and $\boldsymbol{M}$ is g–positive, from (3.31) we obtains

$$\Phi_g(\boldsymbol{U}') - \Phi_g(\boldsymbol{U}) = [\boldsymbol{M}\boldsymbol{U}_0, \boldsymbol{U}_0]_g \geq 0, \tag{3.32}$$

that is

$$\Phi_g(\boldsymbol{U}) \leq \Phi_g(\boldsymbol{U}'), \tag{3.33}$$

for every $\boldsymbol{U}' \in \mathcal{U}$. □

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