LIBERTAS MATHEMATICA, VOL XXI (2001)

ABOUT SOME NEW CIRCUIT PROPERTIES FOR AN UNORIENTED MATROID

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Abstract. Having as start point the circuits C_1, $C_2 \in \mathbf{C}$ of an unoriented matroid $M = (S, \mathbf{F})$, this work establishes some new **properties** of the rank $r(C_1 \cap C_2)$, through the possibilities as the set $C_1 \cap C_2$ could be: independent, dependent, base, circuit, or none of them.

AMS Classification: 05B35.

This paper is realting to unoriented matroids, denoted by $M = (S, \mathbf{F})$, as in [3] or [4]. We know that the *circuits* of a matroid are *minimal dependent sets* C, which determine a class $\mathbf{C}$. The *rank* of a circuit C is defined in [3] or [4], by the relation:

$$r(C) = |C| - 1, \qquad C \in \mathbf{C}. \tag{1}$$

From the same books, we know that, there are matroids as $U_{2,4}$, in which some sets are neither *circuits* and nor *bases* (namely *maximal independent sets*). One of the most interesting axiomatic system, for a matroid, is given by circuits. More exactly, the system was introduced since 1935, in [5], by H. Whitney, the father of the quoted Combinatorics domain. Also, the famous known representant of this new theory, W.T. Tutte, considers that the *circuit axioms* permitted the development of the Matroids, by means of the Graphs.

From [3] and [4], with a new proof in [2], we know the following property:

Proposition 1. *Let $M = (S, \mathbf{F})$ be a matroid, having $\mathbf{C}$ as a circuit class. If C_1, $C_2 \in \mathbf{C}$ so that $C_1 \neq C_2$, we can neither have $C_1 \subset C_2$, nor $C_2 \subset C_1$.*

The *rank-function* r, of a matroid $M = (S, \mathbf{F})$, was also introduced by Whitney, in his pioneer's work [5]. Today, the *rank axioms*, for a matroid, are very usual (see [3] and [4]). The most important properties of the matroid rank, proved in detail, and based on the *inclusion-exclusion Combinatorics principle*, there are in [1], and they will be presented in the:

Theorem 1. *Let $M = (S, \mathbf{F})$ be a matroid and r its rank-function. Then, the following assertions are tru:*

(*i*) $0 \leq r(X) \leq |X|$, $(\forall) X \subseteq S$;
(*ii*) $X \subseteq Y$ *imposes* $r(X) \leq r(Y)$, $(\forall) X, Y \subseteq S$;
(*iii*) $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$, $(\forall) X, Y \subseteq S$.

With all these anterior facts, we could introduce our *new results*, in the following rows.

Proposition 2. *Let $M = (S, \mathbf{F})$ be a matroid, having $\mathbf{C}$ as circuit class. If C_1, $C_2 \in \mathbf{C}$, $C_1 \neq C_2$, then, the lower relation is true:*

$$r(C_1 \cap C_2) \leq \min\{|C_1| - 1, |C_2| - 1\}. \tag{2}$$

PROOF. With the Proposition 1, we can only have $C_1 \cap C_2 \subset C_1$, or $C_1 \cap C_2 \subset C_2$. Now, using (ii) from the Theorem, it results:

$$r(C_1 \cap C_2) \leq r(C_1), \qquad \text{or} \qquad r(C_1 \cap C_2) \leq r(C_2).$$

Because C_1 and C_2 are circuits, with (1), we find:

$$r(C_1 \cap C_2) \leq |C_1| - 1, \qquad \text{or} \qquad r(C_1 \cap C_2) \leq |C_2| - 1,$$

which imediately implies the relation (2). □

Remarks. In the following, we raise for discussion the fact that the intersection $C_1 \cap C_2$ could be dependent or independent, circuit, base, or neither circuit, nor base.

1. We notice that $C_1 \cap C_2$ could not be a dependent set. Really, because $C_1 \cap C_2 \subset C_1$, or $C_1 \cap C_2 \subset C_2$, but C_1, C_2 as circuits, are minimal dependent sets!.
2. Supposing that $C_1 \cap C_2$ could be a circuit, it would have with necessity, as $C_1 \cap C_2 \equiv C_1$, or $C_1 \cap C_2 \equiv C_2$, which implies either $C_1 \equiv C_2$ (in discord with the hypothesis $C_1 \neq C_2$), or $C_1 \subset C_2$, or $C_2 \subset C_1$ (in discord with the Proposition 1).
3. If $C_1 \cap C_2$ is base, then $r(C - 1 \cap C_2) = |C_1 \cap C_2|$, hence:

   $$r(C_1 \cap C_2) < |C_1|, \quad \text{or} \quad r(C_1 \cap C_2) \leq |C_1| - 1 \qquad (\text{and similarly, } r(C_1 \cap C_2) \leq |C_2| - 1).$$

   Supposing that $r(C_1 \cap C_2) = |C_1| - 1$, or $r(C_1 \cap C_2) = |C_2| - 1$, it follows:

   $$|C_1| = |C_1 \cap C_2| + 1, \qquad \text{or} \qquad |C_2| = |C_1 \cap C_2| + 1,$$

   relations which can be transformed through the *inclusion-exclusion principle*, in:

   $$|C_1| + 1 = |C_1 \cup C_2|, \qquad \text{or} \qquad |C_2| + 1 = |C_1 \cup C_2|. \tag{3}$$

   From the remained part, containing strict inequalities, with the same *inclusion-exclusion principle*, and imposing the condition $r(C_1 \cap C_2) < |C_1| - 1$, or $r(C_1 \cap C_2) < |C_2| - 1$, we have:

   $$labeleq : 4r(C_1 \cap C_2) = |C_1 \cap C_2| = |C_1| + |C_2| - |C_1 \cup C_2| < |C_1| - 1 \quad (\text{or}, \ < |C_2| - 1), \tag{4}$$

   from where:

   $$|C_1| + 1 < |C_1 \cup C_2|, \quad \text{or} \quad |C_2| + 1 < |C_1 \cup C_2|. \tag{5}$$

   With (3) and (5), it occurs:

   $$|C_1| + 1 \leq |C_1 \cap C_2|, \quad \text{or} \quad |C_2| + 1 \leq |C_1 \cap C_2|. \tag{6}$$

4. If $C_1 \cap C_2$ is independent set, the things are similar with 3, because $r(C_1 \cap C_2) \leq |C_1 \cap C_2|$. So:

$$r(C_1 \cap C_2) \leq |C_1 \cap C_2| < |C_1|, \text{ hence } r(C_1 \cap C_2) \leq |C_1| - 1, \text{ or } r(C_1 \cap C_2) \leq |C_2| - 1,$$

and as above, we reach the same relation (6).

5. We continue our study, in the hypothesis that $C_1 \cap C_2$ is neither circuit, nor base. Also, with the *inclusion-exclusion principle*, we have:

$$r(C_1 \cap C_2) < |C_1 \cap C_2| = |C_1| + |C_2| - |C_1 \cup C_2|. \tag{7}$$

In (7), we shall impose the conclusion of the Proposition 2, and so, it occurs:

$$|C_1| + 1 < |C_1 \cup C_2|, \quad \text{or} \quad |C_2| + 1 < |C_1 \cup C_2|, \tag{8}$$

as a strict inequality in comparison with (6).

From (6) and (8), we should obtain a finer analysis, if we have in view that $C_1 \cup C_2$ could be base, circuit, or neither base, nor circuit. By example, if $C_1 \cup C_2$ is base, (8) gives us the relation:

$$r(C_1) + 2 < r(C_1 \cup C_2), \quad \text{or} \quad r(C_2) + 2 < r(C_1 \cup C_2), \tag{9}$$

and the relation:

$$r(C_1) + 1 < r(C_1 \cup C_2), \quad \text{or} \quad r(C_2) + 1 < r(C_1 \cup C_2), \tag{10}$$

is obtained if $C_1 \cup C_2$ is circuit.

Also, in the situation in which $C_1 \cup C_2$ is neither base, nor circuit, with the *inclusion-exclusion principle*, from (8) we find:

$$|C_1 \cap C_2| < r(C_1), \quad \text{or} \quad |C_1 \cap C_2| < r(C_2). \tag{11}$$

Taking consideration of all these anterior remarks, we could now emphasize the following *new result*:

Proposition 3. *Let $M = (S, \mathbf{F})$ be a matroid, having $\mathbf{C}$ as circuit class, and $C_1, C_2 \in \mathbf{C}$, $C_1 \neq C_2$ for which the relation (2) functions as in the Proposition 2. The next assertions are true:*

(*i*) *if $C_1 \cap C_2$ is base in M, or if $C_1 \cap C_2$ is independent set in M, then:*

$$|C_1 \cup C_2| \geq \max\{1 + |C_1|, 1 + |C_2|\}; \tag{12}$$

(*ii*) *if $C_1 \cap C_2$ is neither circuit, nor base in M, then:*

$$|C_1 \cup C_2| > \max\{1 + |C_1|, 1 + |C_2|\}. \tag{13}$$

For (3) and (13) we can obtain many other corollaries, with an analysis on $C_1 \cup C_2$ as base, circuit, neither base nor circuit, in the matroid M, as in the relations (9), (10), (11).

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