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# BOUNDARY VALUE PROBLEMS FOR NONCONVEX VALUED DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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**Abstract.** In this paper we investigate the existence of solutions on a compact interval to second order boundary value problems for a class of differential inclusions with nonlocal conditions. We shall rely on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler and on the Schaefer's theorem combined with lower semicontinuous multivalued operators with decomposable values.

**Keywords:** Boundary value problems, nonlocal conditions, measurable selection, contraction multivalued map, existence, fixed point, decomposable valued map.
**AMS (MOS) Subject Classifications:** 34B10, 34B15.

## 1 Introduction

In this paper we shall prove existence of solutions defined on a compact real interval for the boundary value problem (BVP for short) of the second order differential inclusion with nonlocal boundary conditions

$$y'' \in F(t,y), \quad t \in J = [0,1] \tag{1.1}$$

$$\begin{cases} y(0) + f_0(y) = c_0 \\ y(1) + f_1(y) = c_1 \end{cases} \tag{1.2}$$

where $F : J \times \mathbb{R}^n \longrightarrow \mathcal{P}(\mathbb{R}^n)$ is a given multivalued map satisfying some assumptions that will be specified later, $f_0, f_1 \in C(C(J,\mathbb{R}^n),\mathbb{R}^n)$ $c_i \in \mathbb{R}^n, i = 1,2$ and $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets in $\mathbb{R}^n$.

The BVP problem (1.1)-(1.2) in the case when the multivalued $F$ has convex values was studied by Benchohra and Ntouyas in [1], by using a fixed point theorem for condensing multivalued maps due to Martelli. Here we consider the case when the multivalued $F$ has nonconvex values. By using the fixed point theorem for contraction maps due to Covitz and Nadler [12], and the Schaefer's theorem combined with a selection theorem of Bressan and Colombo for lower semicontinuous multivalued operators with decomposable values, existence results are proposed for the problem (1.1)-(1.2).

Initial value problems with nonlocal conditions were studied on compact intervals by several authors, see e.g. [10], [23] for single valued differential equations and on infinite intervals by [3] and by [2], [4]–[6] for differential inclusions. Nonlocal boundary value problems were investigated e.g. in [21], [22], [16]–[19] for special cases of the functions $f_0$ and $f_1$. For more information we refer the interested reader to the references cited in the above papers.

## 2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

By $C(J, \mathbb{R}^n)$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

$L^1(J, \mathbb{R}^n)$ denotes the Banach space of measurable functions $y : J \longrightarrow E$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_0^1 |y(t)|\, dt \quad \text{for all} \quad y \in L^1(J, \mathbb{R}^n).$$

$AC^i([0,1], \mathbb{R}^n)$ is the space of $i$-times differentiable functions $y : [0,1] \to \mathbb{R}^n$, whose $i^{th}$ derivative, $y^{(i)}$, is absolutely continuous.
Let $A$ be a subset of $J \times \mathbb{R}^n$. $A$ is $\mathcal{L}\otimes\mathcal{B}$ measurable if $A$ belongs to the $\sigma$-algebra generated by all sets of the form $\mathcal{J} \times D$ where $\mathcal{J}$ is Lebesgue measurable in $J$ and $D$ is Borel measurable in $\mathbb{R}^n$. A subset $B$ of $L^1(J, \mathbb{R}^n)$ is decomposable if, for all $u, v \in B$ and $\mathcal{J} \subset J$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in B$, where $\chi$ denotes for the characteristic function.

Let $E$ be a Banach space, $X$ a nonempty closed subset of $E$ and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. $G$ is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap C \neq \emptyset\}$ is open for any open set $C$ in $E$. $G$ has a fixed point if there is $x \in X$ such that $x \in G(x)$.

**Definition 2.1.** *Let $Y$ be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J, \mathbb{R}^n))$ be a multivalued operator. We say $N$ has property (BC) if*

*1) $N$ is lower semi-continuous (l.s.c.);*
*2) $N$ has nonempty closed and decomposable values.*

Let $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. Assign to $F$ the multivalued operator

$$\mathcal{F} : C(J, \mathbb{R}^n) \to \mathcal{P}(L^1(J, \mathbb{R}^n))$$

by letting

$$\mathcal{F}(y) = \{w \in L^1(J, \mathbb{R}^n) : w(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with $F$. We say $F$ is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

**Lemma 2.2.** *[9] Let $Y$ be separable metric space and let $N : Y \to \mathcal{P}(L^1(J, \mathbb{R}^n))$ be a multivalued operator which has property (BC). Then $N$ has a continuous selection, i.e. there exists a continuous function (single-valued) $g : Y \to L^1(J, \mathbb{R}^n)$ such that $g(y) \in N(y)$ for every $y \in Y$.*

Let $(X, d)$ be a metric space. We use the notations:

$P(X) = \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$.

Consider $H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

**Definition 2.3.** *A multivalued operator $N : X \to P_{cl}(X)$ is called*

*a) $\gamma$-Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x,\ y \in X,$$

*b) contraction if and only if it is $\gamma$-Lipschitz with $\gamma < 1$.*

For more details on multivalued maps and the proof of known results cited in this section we refer to the books of Deimling [13], Gorniewicz [15], Hu and Papageorgiou [20] and Tolstonogov [25].

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [12] (see also Deimling, [13] Theorem 11.1).

**Lemma 2.4.** *Let $(X, d)$ be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

## 3 Main Result

In the next Theorem we give our first existence result for the BVP (1.1)–(1.2).

**Definition 3.1.** *A function $y \in AC^1(J, \mathbb{R}^n)$ is called solution for the BVP (1.1)-(1.2) if $y$ satisfies the differential inclusion (1.1) a.e. on $J$ and the condition (1.2).*

**Theorem 3.2.** *Assume that:*

*(H1) $F : J \times \mathbb{R}^n \longrightarrow P_{cl}(\mathbb{R}^n)$ has the property that $F(\cdot, u) : J \to P_{cl}(\mathbb{R}^n)$ is measurable for each $u \in \mathbb{R}^n$;*

*(H2) There exists $l \in L^1(J, \mathbb{R})$ such that*

$$H_d(F(t, y), F(t, \overline{y})) \leq l(t)|y - \overline{y}|, \text{ for each } t \in J \text{ and } y, \overline{y} \in \mathbb{R}^n,$$

*and*

$$d(0, F(t, 0)) \leq l(t), \quad \text{for almost each} \quad t \in J.$$

*(H3)* $|f_i(y) - f_i(\overline{y})| \leq c_i\|y_1 - y_2\|_\infty$, *$i = 0, 1$ for each $t \in J$ and $y, \overline{y} \in C(J, \mathbb{R}^n)$, where $c_i, i = 0, 1$ are nonnegative constants.*

*Then the IVP (1.1)-(1.2) has at least one solution on $J$.*

**Proof.** We transform the problem (1.1)-(1.2) into a fixed point problem. Consider the multivalued operator, $N : C(J, \mathbb{R}^n) \to \mathcal{P}(C(J, \mathbb{R}^n))$ defined by:

$$N(y) := \Big\{ h \in C(J, \mathbb{R}^n) : h(t) = c_0 - f_0(y) + [c_1 - c_0 - f_1(y) + f_0(y)]t + \int_0^1 G(t,s)g(s)ds, \ t \in J \Big\}$$

where $G$ is the Green's function for the BVP

$$y''(t) = 0, \quad y(0) = 0, \quad y(1) = 0$$

which is given by the formula

$$G(x,s) = \begin{cases} (1-x)s, & \text{if } 0 \leq s \leq x \leq 1 \\ (1-s)x, & \text{if } 0 \leq x \leq s \leq 1 \end{cases}$$

and

$$g \in S_{F,y} = \Big\{ g \in L^1(J, \mathbb{R}^n) : g(t) \in F(t, y(t)) \ \text{ for a.e. } \ t \in J \Big\}.$$

We shall show that $N$ satisfies the assumptions of Lemma 2.4. The proof will be given in two steps.

**Step 1:** *$N(y) \in P_{cl}(C(J, \mathbb{R}^n))$ for each $y \in C(J, \mathbb{R}^n)$.*

Indeed, let $(y_n)_{n \geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in $C(J, \mathbb{R}^n)$. Then $\tilde{y} \in C(J, \mathbb{R}^n)$ and

$$y_n(t) \in c_0 - f_0(y_1) + [c_1 - c_0 - f_1(y_1) + f_0(y_1)]t + \int_0^1 G(t,s)F(s, y(s))ds, \ t \in J.$$

Using the closedness property of the values of $F$ and the second part of $(H2)$ we can prove that $\int_0^1 G(t,s)F(s, y(s))(s)ds$ is closed for each $t \in J$. Then

$$\begin{aligned} & y_n(t) \longrightarrow \tilde{y}(t) \in \\ & \qquad c_0 - f_0(y_1) + [c_1 - c_0 - f_1(y_1) + f_0(y_1)]t + \int_0^1 G(t,s)F(s, y(s))ds, \ t \in J. \end{aligned}$$

So $\tilde{y} \in N(y)$.

**Step 2:** *$H_d(N(y_1), N(y_2)) \leq \gamma\|y_1 - y_2\|_\infty$ for each $y_1, y_2 \in C(J, E)$ (where $\gamma < 1$).*

Let $y_1, y_2 \in C(J, \mathbb{R}^n)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t, y_1(t))$ such that

$$h_1(t) = c_0 - f_0(y_1) + [c_1 - c_0 - f_1(y_1) + f_0(y_1)]t + \int_0^1 G(t,s)g_1(s)ds, \quad t \in J.$$

From (H2) it follows that

$$H_d(F(t, y_1(t)), F(t, y_2(t))) \leq l(t)|y_1(t) - y_2(t)|.$$

Hence there is $w \in F(t, y_2(t))$ such that

$$|g_1(t) - w| \leq l(t)|y_1(t) - y_2(t)|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R}^n)$, given by

$$U(t) = \{w \in \mathbb{R}^n : |g_1(t) - w| \leq l(t)|y_1(t) - y_2(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_2(t))$ is measurable (see Proposition III.4 in [11]) there exists $g_2(t)$ a measurable selection for $V$. So, $g_2(t) \in F(t, y_2(t))$ and

$$|g_1(t) - g_2(t)| \leq l(t)|y_1(t) - y_2(t)|, \quad \text{for each} \quad t \in J.$$

Let us define for each $t \in J$

$$h_2(t) = c_0 - f_0(y_2) + [c_1 - c_0 - f_1(y_2) + f_0(y_2)]t + \int_0^1 G(t, s) g_2(s) ds.$$

Then we have

$$\begin{aligned}
|h_1(t) - h_2(t)| &\leq |f_0(y_1) - f_0(y_2)| + |f_1(y_1) - f_1(y_2)| \\
&\quad + \int_0^1 G(t, s) \|g_1(s) - g_2(s)\| \, ds \\
&\leq (c_0 + c_1) \|y_1 - y_2\|_\infty \\
&\quad + \sup_{J \times J} |G(t, s)| \int_0^1 l(s)|y_1(s) - y_2(s)| ds \\
&\leq \left[ c_0 + c_1 + \sup_{J \times J} |G(t, s)| \int_0^1 l(t) dt \right] \|y_1 - y_2\|_\infty.
\end{aligned}$$

Then

$$\|h_1 - h_2\|_\infty \leq \left[ c_0 + c_1 + \sup_{J \times J} |G(t, s)| \int_0^1 l(t) dt \right] \|y_1 - y_2\|_\infty.$$

By the analogous relation, obtained by interchanging the roles of $y_1$ and $y_2$, it follows that

$$H_d(N(y_1), N(y_2)) \leq \left[ c_0 + c_1 + \sup_{J \times J} |G(t, s)| \int_0^1 l(t) dt \right] \|y_1 - y_2\|_\infty.$$

So, if we choose $c_0 + c_1 + \sup_{J \times J} |G(t, s)| \int_0^1 l(t) dt < 1$, $N$ is a contraction, and thus, by Lemma 2.2, it has a fixed point $y$, which is solution to (1.1)-(1.2).

By the help of the Schaefer's theorem, combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values, we shall present an existence result for the problem $(1.1) - -(1.2)$. Before this, let us introduce the following hypotheses which are assumed hereafter:

(H4) $F : [0,1] \times \mathbb{R}^n \longrightarrow \mathcal{P}(\mathbb{R}^n)$ is a nonempty compact valued multivalued map such that
  a) $(t, y) \mapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
  b) $y \mapsto F(t, y)$ is lower semi-continuous for a.e. $t \in [0, 1]$;

(H5) $F$ is integrably bounded, that is, there exists a function $H \in L^1([0,1], \mathbb{R}^+)$ such that

$$\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq H(t) \text{ for a.e. } t \in [0,1] \text{ and } y \in \mathbb{R}^n.$$

In the proof of our following theorem we will need to the auxiliary result:

**Lemma 3.3.** *[14]. Let $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty, compact values. Assume $(H4)$ and $(H5)$ hold. Then $F$ is of l.s.c. type.*

**Theorem 3.4.** *Suppose hypotheses (H4), (H5) and the following hold:*

*(H6) The functions $f_i, i = 0, 1$ are completely continuous and there are constants $k_i, i = 1, 2$ such that*

$$|f_i(y)| \leq k_i, i = 0, 1 \quad \textit{for all} \quad y \in C(J, \mathbb{R}^n).$$

*Then the boundary value problem (1.1)–(1.2) has at least one solution.*

**Proof.** (H4) and (H5) imply by Lemma 3.3 that $F$ is of lower semi-continuous type. Then from Lemma 2.2 there exists a continuous function $g : C(J, \mathbb{R}^n) \to L^1(J, \mathbb{R}^n)$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in C(J, \mathbb{R}^n)$.

We consider the problem

$$y''(t) = g(y)(t), \quad \text{a.e. } t \in J = [0, 1], \tag{3.1}$$

$$y(0) + f_0(y) = c_0, \ y(1) + f_1(y) = c_1. \tag{3.2}$$

**Remark 3.5.** If $y \in C(J, \mathbb{R}^n)$ is a solution of the problem (3.1)–(3.2), then $y$ is a solution to the problem (1.1)–(1.2).

Transform problem (3.1)–(3.2) into a fixed point problem. Consider the multivalued map, $\overline{N} : C(J, \mathbb{R}^n) \longrightarrow C(J, \mathbb{R}^n)$ defined by:

$$\overline{N}(y)(t) = c_0 - f_0(y) + [c_1 - c_0 - f_1(y) + f_0(y)]t + \int_0^1 G(t, s) g(y)(s) ds$$

Clearly from (H5)–(H6) and the Arzela-Ascoli theorem the multivalued operator $\overline{N}$ is continuous and completely continuous.

In order to apply Schaefer's theorem, it remains to show that the set

$$\mathcal{E}(\overline{N}) := \{y \in C(J, \mathbb{R}^n) : \lambda y = \overline{N}(y), \text{ for some } \lambda > 1\}$$

is bounded. Let $y \in \mathcal{E}(\overline{N})$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus for each $t \in J$

$$y(t) = \lambda^{-1}[c_0 - f_0(y)] + \lambda^{-1}[c_1 - c_0 - f_1(y) + f_0(y)]t + \lambda^{-1} \int_0^1 G(t, s) g(y)(s) ds.$$

This implies by (H5) and (H6) that for each $t \in J$ we have

$$|y(t)| \leq 2|c_0| + |c_1| + 2k_0 + k_1 + \int_0^1 |G(t,s)|H(s)ds.$$

Thus

$$\|y\|_\infty \leq 2|c_0| + |c_1| + 2k_0 + k_1 + \sup_{(t,s)\in J\times J} |G(t,s)| \int_0^1 H(s)ds.$$

This shows that $\Omega$ is bounded.

As a consequence of Schaefer's theorem (see [24] p. 29) we deduce that $\overline{N}$ has a fixed point which is a solution of (3.1)–(3.2) and hence from Remark 3.5 a solution to the problem (1.1)–(1.2).

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