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ON SOME NEW ESTIMATES OF THE FINITE HILBERT TRANSFORM

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Abstract. Some estimates for the finite Hilbert transform via the mid-point inequality are given.

1 Introduction

Denote by $T(f)(a,b,\cdot)$ the finite *Hilbert transform* of the function $f:(a,b)\to\mathbb{R}$, i.e., we recall it

$$\begin{aligned} T(f)(a,b;t) &= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)}{\tau-t}d\tau \\ &:= \lim_{\varepsilon\downarrow 0}\left[\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right]\left(\frac{f(\tau)}{\pi(\tau-t)}d\tau\right), \end{aligned} \tag{1.1}$$

where PV has the usual meaning of the *Cauchy principle value.*

In [7], by the use of trapezoidal type inequalitites, the authors pointed out the following estimation results for the Hilbert transform of functions whose derivative $f':(a,b)\to\mathbb{R}$ are absolutely continuous on (a,b).

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be such that $f':(a,b)\to\mathbb{R}$ is absolutely continuous on (a,b). Then we have the bounds:*

$$\begin{aligned} &\left|T(f)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'(t)(b-a)\right]\right| \\ &\leq\begin{cases} \dfrac{\|f''\|_\infty}{4\pi}\left[\dfrac{(b-a)^2}{4}+\left(t-\dfrac{a+b}{2}\right)^2\right] & \text{if } f''\in L_\infty[a,b]; \\ \dfrac{q\|f''\|_p}{4\pi(q+1)^{1+\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right] & \text{if } f''\in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{2\pi}\|f''\|_1(b-a), \end{cases} \end{aligned} \tag{1.2}$$

for any $t\in(a,b)$.

If one would assume more about the function f, then the following result also holds [7]:

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f'':(a,b)\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then:*

$$\left|T(f)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'(t)(b-a)\right]\right| \tag{1.3}$$
$$\leq\begin{cases}\dfrac{\|f''\|_\infty}{12\pi}(b-a)\left[\dfrac{(b-a)^2}{12}+\left(t-\dfrac{a+b}{2}\right)^2\right] & \text{if } f''\in L_\infty[a,b];\\[2ex] \dfrac{q\|f''\|_p\left[B(q+1,q+1)\right]^{\frac{1}{q}}}{2\pi(2q+1)}\left[(t-a)^{2+\frac{1}{q}}+(b-t)^{2+\frac{1}{q}}\right] & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{\|f''\|_1}{8\pi}\left[\dfrac{(b-a)^2}{4}+\left(t-\dfrac{a+b}{2}\right)^2\right],\end{cases}$$

for any $t\in(a,b)$.

For classical results on the finite Hilbert transform, see [1]–[5]. In [6], the reader may find different inequalities for monotonic or convex functions whose Hilbert transforms are supposed to be defined in every point of the open interval (a,b).

In the present paper, by the use of midpoint type inequalities, we point out different approximates for the finite Hilbert transform. Some numerical experiments are also provided.

2 The Results

The following result holds.

Theorem 2.1. *Assume that the function $f:[a,b]\to\mathbb{R}$ is such that $f':[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\left|T(f)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right]\right| \tag{2.1}$$
$$\leq\begin{cases}\dfrac{\|f''\|_\infty}{4\pi}\left[\left(t-\dfrac{a+b}{2}\right)^2+\dfrac{(b-a)^2}{4}\right] & \text{if } f''\in L_\infty[a,b];\\[2ex] \dfrac{q\|f''\|_p}{2\pi(q+1)^{1+\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right] & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{\|f''\|_1}{2\pi}(b-a),\end{cases}$$
$$\leq\begin{cases}\dfrac{\|f''\|_\infty}{8\pi}(b-a)^2 & \text{if } f''\in L_\infty[a,b];\\[2ex] \dfrac{q\|f''\|_p}{\pi(q+1)^{1+\frac{1}{q}}}(b-a)^{1+\frac{1}{q}} & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{1}{2\pi}\|f''\|_1(b-a),\end{cases}$$

for any $t \in (a,b)$. *The* $\|\cdot\|_p$, $p \in [1,\infty]$ *denote the usual norms, i.e.,*

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)| \quad if \;\; g \in L_\infty[a,b]$$

and

$$\|g\|_p := \left(\int_a^b |g(t)|^p \, dt\right)^{\frac{1}{p}} \quad if \;\; g \in L_p[a,b], \; p \geq 1.$$

PROOF: As for the mapping $f_0 : (a,b) \to \mathbb{R}$, $f_0(t) = 1$, $t \in (a,b)$, we have

$$T(f_0)(a,b;t) = \frac{1}{\pi} \ln\left(\frac{b-t}{t-a}\right), \; t \in (a,b),$$

then, obviously

$$\begin{aligned}(Tf)(a,b;t) &= \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau \\ &= \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + \frac{f(t)}{\pi} PV \int_a^b \frac{d\tau}{\tau - t},\end{aligned}$$

from where we get the identity (see also [7]):

$$(Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau. \tag{2.2}$$

If we use the known identity, which can easily be proved using the integration by parts formula,

$$\int_\alpha^\beta g(u)\, du = g\left(\frac{\alpha+\beta}{2}\right)(\beta - \alpha) + \int_\alpha^\beta K(u)\, g'(u)\, du, \tag{2.3}$$

where

$$K(u) := \begin{cases} u - \alpha \text{ if } u \in \left[\alpha, \frac{\alpha+\beta}{2}\right] \\ \\ u - \beta \text{ if } u \in \left(\frac{\alpha+\beta}{2}, \beta\right] \end{cases}$$

and g is absolutely continuous on $[a,b]$, we may write

$$\begin{aligned} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau &= PV \int_a^b \frac{\int_t^\tau f'(u)\, du}{\tau - t} d\tau \\ &= PV \int_a^b \left[\frac{f'\left(\frac{\tau+t}{2}\right)(\tau - t) + \int_t^\tau K(u)\, f''(u)\, du}{\tau - t}\right] d\tau \\ &= PV \int_a^b f'\left(\frac{\tau+t}{2}\right) d\tau + PV \int_a^b \left(\frac{1}{\tau - t} \int_t^\tau K(u)\, f''(u)\, du\right) d\tau \\ &= 2\left[f\left(\frac{b+t}{2}\right) - f\left(\frac{a+t}{2}\right)\right] + PV \int_a^b \left(\frac{1}{\tau - t} \int_t^\tau K(u)\, f''(u)\, du\right) d\tau. \end{aligned}$$

Consequently, by (2.2), we obtain the identity

$$\begin{aligned}&(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right] \qquad (2.4)\\ &=\frac{1}{\pi}PV\int_a^b\left(\frac{1}{\tau-t}\int_t^\tau K(u)\,f''(u)\,du\right)d\tau,\end{aligned}$$

where

$$K(u)=\begin{cases}u-t \text{ if } u\in\left[t,\frac{\tau+t}{2}\right]\\ u-\tau \text{ if } u\in\left(\frac{\tau+t}{2},\tau\right]\end{cases}.$$

Using the properties of modulus, we get, by (2.4), that

$$\begin{aligned}&\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right]\right| \qquad (2.5)\\ &\leq\frac{1}{\pi}PV\int_a^b\left|\frac{1}{\tau-t}\int_t^\tau K(u)\,f''(u)\,du\right|d\tau=:D(a,b;t).\end{aligned}$$

Now, it is obvious that

$$\begin{aligned}\left|\int_t^\tau K(u)\,f''(u)\,du\right| &\leq \sup_{u\in[a,b]}|f''(u)|\left|\int_t^\tau K(u)\,du\right|\\ &=\|f''\|_\infty\left|\int_t^{\frac{\tau+t}{2}}(u-t)\,du+\int_{\frac{\tau+t}{2}}^{\tau}(t-u)\,du\right|\\ &=\|f''\|_\infty\frac{(t-\tau)^2}{4}.\end{aligned}$$

Then

$$\begin{aligned}D(a,b;t) &\leq \frac{1}{4\pi}\|f''\|_\infty PV\int_a^b|t-\tau|\,d\tau\\ &=\frac{\|f''\|_\infty}{4\pi}\cdot\frac{(t-a)^2+(b-t)^2}{2}\\ &=\frac{\|f''\|_\infty}{4\pi}\left[\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{4}\right].\end{aligned}$$

Using Hölder's integral equality, we have

$$\begin{aligned}\left|\int_t^\tau K(u)\,f''(u)\,du\right| &\leq \left|\int_t^\tau|f''(u)|^p\,du\right|^{\frac{1}{p}}\left|\int_t^\tau|K(u)|^q\,du\right|^{\frac{1}{q}}\\ &\leq\|f''\|_p\left|\int_t^\tau|K(u)|^q\,du\right|^{\frac{1}{q}}\\ &=\|f''\|_p\left|\int_t^{\frac{\tau+t}{2}}(u-t)^q\,du+\int_{\frac{\tau+t}{2}}^{\tau}(t-u)^q\,du\right|^{\frac{1}{q}}\\ &=\|f''\|_p\left[\frac{|\tau-t|^{q+1}}{2^q(q+1)}\right]^{\frac{1}{q}}=\frac{\|f''\|_p\,|t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\end{aligned}$$

for all $t,\tau\in(a,b)$.

Then

$$D(a,b;t) \le \frac{1}{\pi}\left\|f''\right\|_p PV\int_a^b \frac{|t-\tau|^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}d\tau$$
$$= \frac{q\left\|f''\right\|_p\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]}{2\pi(q+1)^{1+\frac{1}{q}}}$$

proving the second part of the first inequality in (2.1).

Finally, we observe that

$$\left|\int_t^\tau K(u)f''(u)\,du\right| \le \sup_{u\in[t,\tau]}|K(u)|\left|\int_t^\tau |f''(u)|\,du\right|$$
$$= \frac{\left\|f''\right\|_1}{2\pi}|t-\tau|$$

and then

$$D(a,b;t) \le \frac{1}{2\pi}\left\|f''\right\|_1 PV\int_a^b d\tau = \frac{1}{2\pi}\left\|f''\right\|_1(b-a),$$

proving the last part of the second inequality in (2.1).

The last part of (2.1) is obvious. □

The best inequality we can get from (2.1) is embodied in the following corollary.

Corollary 2.2. *With the assumptions in Theorem 2.1, we have*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\frac{2}{\pi}\left[f\left(\frac{a+3b}{4}\right)-f\left(\frac{3a+b}{4}\right)\right]\right|$$
$$\le \begin{cases} \frac{1}{16\pi}\left\|f''\right\|_\infty(b-a)^2 & \text{if } f''\in L_\infty[a,b]; \\ \frac{q}{2^{1+\frac{1}{q}}\pi(q+1)^{1+\frac{1}{q}}}\left\|f''\right\|_p(b-a)^{1+\frac{1}{q}} & \text{if } f''\in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases} \tag{2.6}$$

Remark. It is also obvious that if $b-a\to 0$, then both the inequalities (2.1) and (2.6) provide accurate approximations.

The following result holds.

Theorem 2.3. *Assume that the function* $f:[a,b]\to\mathbb{R}$ *is such that* $f''':[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$. *Then we have the inequalities:*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right]\right|\le$$

$$\leq \begin{cases} \dfrac{\|f'''\|_{\infty}}{24\pi}(b-a)\left[\dfrac{(b-a)^2}{12}+\left(t-\dfrac{a+b}{2}\right)^2\right] & \text{if } f''' \in L_{\infty}[a,b]; \\ \dfrac{q\,\|f'''\|_p}{8\pi(2q+1)^{1+\frac{1}{q}}}\left[(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}\right] & \text{if } f''' \in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\|f'''\|_1}{8\pi}\left[\dfrac{(b-a)^2}{4}+\left(t-\dfrac{a+b}{2}\right)^2\right], \end{cases} \tag{2.7}$$

$$\leq \begin{cases} \dfrac{\|f'''\|_{\infty}(b-a)^3}{72\pi} & \text{if } f''' \in L_{\infty}[a,b]; \\ \dfrac{q\,\|f'''\|_p(b-a)^{2+\frac{1}{q}}}{8\pi(2q+1)^{1+\frac{1}{q}}} & \text{if } f''' \in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\|f'''\|_1(b-a)^2}{16\pi}, \end{cases}$$

Proof: If we use the identity (2.2) and the following identity, which can be proved by applying the integration by parts formula twice,

$$\int_{\alpha}^{\beta} g(u)\,du = g\left(\frac{\alpha+\beta}{2}\right)(\beta-\alpha)+\frac{1}{2}\int_{\alpha}^{\beta} L(u)\,g''(u)\,du,$$

where

$$L(u) := \begin{cases} (u-\alpha)^2 \text{ if } u \in \left[\alpha, \frac{\alpha+\beta}{2}\right] \\ (u-\beta)^2 \text{ if } u \in \left(\frac{\alpha+\beta}{2}, \beta\right] \end{cases}$$

and g is such that g' is absolutely continuous on $[\alpha, \beta]$, then we get

$$\begin{aligned} (Tf)(a,b;t) &- \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) \\ &= \frac{1}{\pi}PV\int_a^b\left[\frac{f'\left(\frac{\tau+t}{2}\right)(\tau-t)\frac{1}{2}\int_t^{\tau} L(u)\,f'''(u)\,du}{\tau-t}\right]d\tau \\ &= \frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right]+\frac{1}{2\pi}PV\int_a^b\left[\frac{1}{\tau-t}\int_t^{\tau} L(u)\,f'''(u)\,du\right]d\tau. \end{aligned}$$

Consequently, we have the identity:

$$\begin{aligned} (Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) &- \frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right] \\ &= \frac{1}{2\pi}PV\int_a^b\left[\frac{1}{\tau-t}\int_t^{\tau} L(u)\,f'''(u)\,du\right]d\tau, \end{aligned} \tag{2.8}$$

where

$$L(u) = \begin{cases} (u-t)^2 & \text{if } u \in \left[t, \frac{\tau+t}{2}\right] \\ (u-\tau)^2 & \text{if } u \in \left(\frac{\tau+t}{2}, \tau\right] \end{cases}.$$

Using the modulus properties, we may write, by (2.8), that

$$\begin{aligned} &\left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right) - f\left(\frac{t+a}{2}\right)\right]\right| \\ &\leq \frac{1}{2\pi} PV \int_a^b \left|\frac{1}{\tau - t}\int_t^\tau |L(u)|\,|f'''(u)|\,du\right| d\tau =: E(a,b;t). \end{aligned} \tag{2.9}$$

Now, observe that

$$\begin{aligned} \left|\int_t^\tau |L(u)|\,|f'''(u)|\,du\right| &\leq \|f'''\|_\infty \left|\int_t^{\frac{\tau+t}{2}} (u-t)^2\,du + \int_{\frac{\tau+t}{2}}^\tau (t-u)^2\,du\right| \\ &= \frac{\|f'''\|_\infty}{12}|t-\tau|^3 \end{aligned}$$

and then

$$\begin{aligned} E(a,b;t) &\leq \frac{\|f'''\|_\infty}{24\pi}\int_a^b (t-\tau)^2\,d\tau = \frac{\|f'''\|_\infty}{24\pi}\cdot\frac{(b-t)^3 + (t-a)^3}{3} \\ &= \frac{\|f'''\|_\infty}{24\pi}\left[\frac{(b-a)^2}{12} + \left(t - \frac{a+b}{2}\right)^2\right](b-a), \end{aligned}$$

giving the first part of the first inequality in (2.7).

Using Hölder's inequality, we may write that

$$\begin{aligned} \left|\int_t^\tau |L(u)|\,|f'''(u)|\,du\right| &\leq \|f'''\|_p \left|\int_t^{\frac{\tau+t}{2}} |u-t|^{2q}\,du + \int_{\frac{\tau+t}{2}}^\tau |t-u|^{2q}\,du\right|^{\frac{1}{q}} \\ &= \|f'''\|_p \left[\frac{2\cdot\left|\frac{t-\tau}{2}\right|^{2q+1}}{2q+1}\right]^{\frac{1}{q}} = \frac{1}{4(2q+1)^{\frac{1}{q}}}\|f'''\|_p\,|t-\tau|^{2+\frac{1}{q}} \end{aligned}$$

and then

$$\begin{aligned} E(a,b;t) &\leq \frac{\|f'''\|_p}{8\pi(2q+1)^{\frac{1}{q}}} PV \int_a^b |t-\tau|^{1+\frac{1}{q}}\,d\tau \\ &= \frac{\|f'''\|_p}{8\pi(2q+1)^{\frac{1}{q}}}\cdot\frac{(b-t)^{2+\frac{1}{q}} + (t-a)^{2+\frac{1}{q}}}{\frac{2q+1}{q}} \\ &= \frac{q\,\|f'''\|_p}{8\pi(2q+1)^{\frac{1}{q}}}\left[(b-t)^{2+\frac{1}{q}} + (t-a)^{2+\frac{1}{q}}\right], \end{aligned}$$

which proves the second part of the first inequality in (2.7).

Finally,

$$\left|\int_t^\tau |L(u)|\,|f'''(u)|\,du\right| \le \sup_{u\in[t,\tau]} |L(u)|\,\|f'''\|_1 = \frac{|t-\tau|}{4}\,\|f'''\|_1,$$

giving

$$\begin{aligned} E(a,b;t) &\le \frac{\|f'''\|_1}{8\pi} PV\int_a^b |t-\tau|\,d\tau = \frac{\|f'''\|_1}{8\pi}\cdot\frac{(b-t)^2+(t-a)^2}{2} \\ &= \frac{\|f'''\|_1}{8\pi}\left[\frac{(b-a)^2}{4}+\left(t-\frac{a+b}{2}\right)^2\right], \end{aligned}$$

which proves the last part of the first inequality in (2.7). □

The best inequality we may obtain from (2.7) is embodied in the following corollary.

Corollary 2.4. *With the assumptions of Theorem 2.3, we have*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\frac{2}{\pi}\left[f\left(\frac{a+3b}{4}\right)-f\left(\frac{3a+b}{4}\right)\right]\right| \le \begin{cases} \dfrac{\|f'''\|_\infty (b-a)^3}{288\pi} & \text{if } f'''\in L_\infty[a,b]; \\ \dfrac{q\,\|f'''\|_p (b-a)^{2+\frac{1}{q}}}{16\cdot 2^{\frac{1}{q}}\pi(2q+1)^{\frac{1}{q}}} & \text{if } f'''\in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\|f'''\|_1 (b-a)^2}{32\pi} & \text{if } f'''\in L_1[a,b] \end{cases} \tag{2.10}$$

3 Numerical Experiments

For the function $f:[a,b]\to\mathbb{R}$, we may consider the expression

$$E(f;a,b,t) := \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+\frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right],\quad t\in[a,b].$$

As shown in Theorems 2.1 and 2.3, $E(f;a,b,\cdot)$ provides an approximation for the finite Hilbert transform $(Tf)(a,b,\cdot)$. We remark that for small intervals, i.e., $b-a\to 0$, the approximation is accurate (with the order 2 or 3).

If we consider the function $f:[1,2]\to\mathbb{R}$, $f(t)=\sin t$, then the plots of the exact transform $(Tf)(a,b,\cdot)$ and the approximate $E(f;a,b,\cdot)$ are embodied in Figures 1 and 2.

If the same function $f(t)=\sin t$ is considered on the smaller interval, say $[0.01, 0.02]$, the accuracy of the approximation (see Figures 3 and 4) is much better, as expected from the theoretical results outlined above.

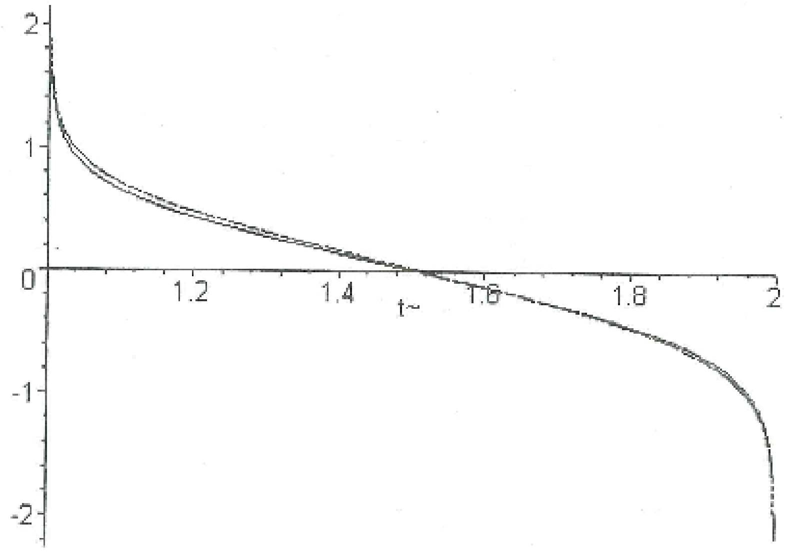

2
1
0
-1
-2
1.2
1.4
1.6
1.8
2
t~

Fig. 1.

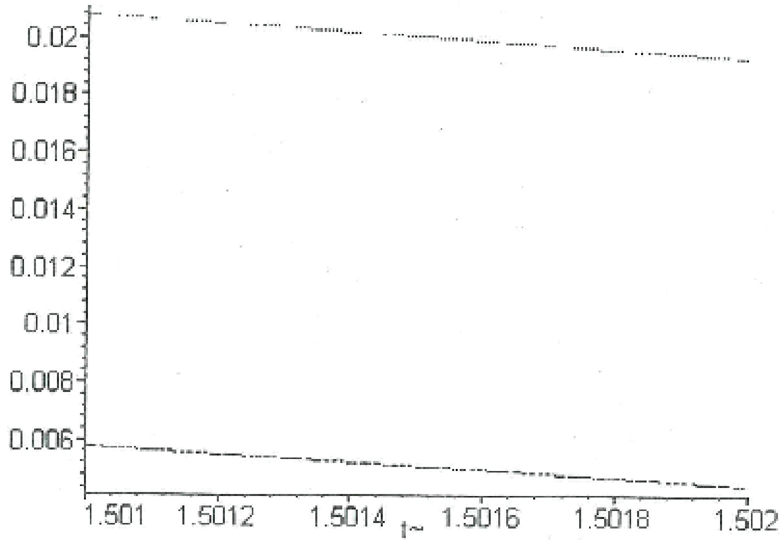
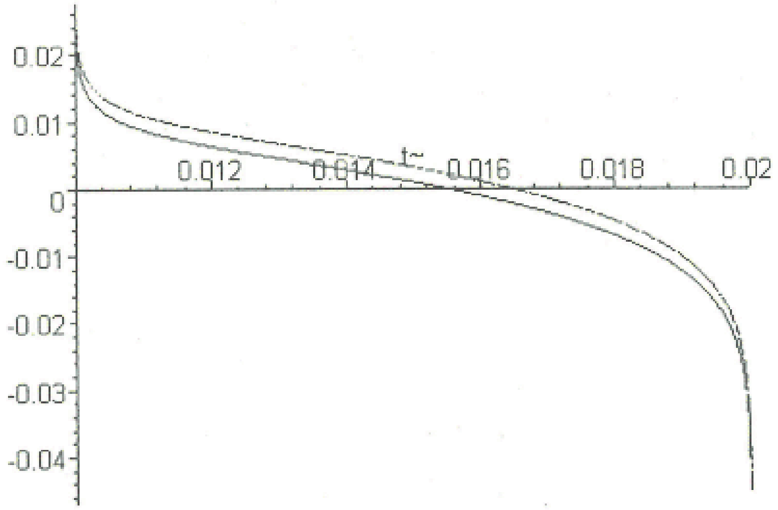

0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
1.501
1.5012
1.5014
1.5016
1.5018
1.502
t~

Fig. 2.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0.01
0
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-0.03
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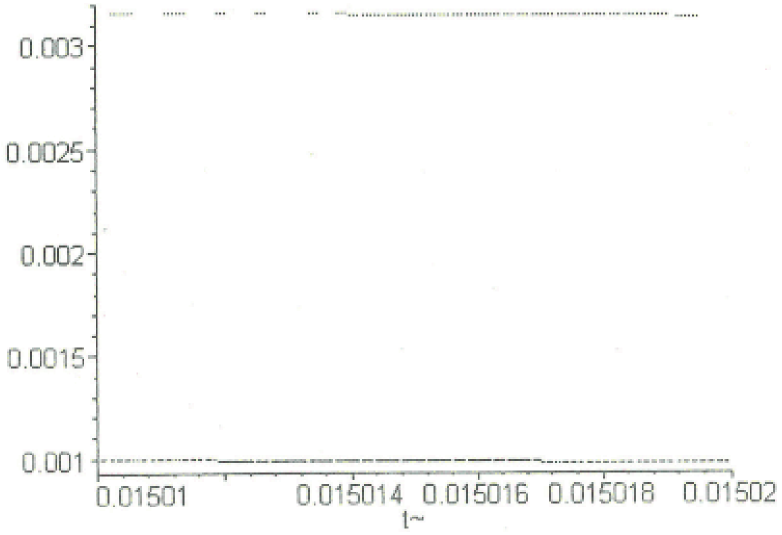
0.012
0.014
0.016
0.018
0.02
t~

Fig. 3.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0.0025
0.002
0.0015
0.001
0.01501
0.015014
0.015016
0.015018
0.01502
t~

Fig. 4.

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