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# ON CERTAIN SUBCLASSES OF STARLIKE FUNCTIONS

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**Abstract.** In this paper we introduce certain subclasses of starlike functions which extend the well-known class of alpha-convex functions (the case $\beta = 0$) and the class of alpha-starlike functions (the case $\beta = 1$).

## 1 Introduction and preliminaries

Let $U$ be the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\},$$

let $\mathcal{H}(U)$ be the space of holomorphic function in $U$.

We let:

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \ z \in U\}$$

with $A_1 = A$,

$$S^* = \left\{ f \in A, \ \text{Re}\, \frac{zf'(z)}{f(z)} > 0, \ z \in U \right\},$$

the class of starlike functions in $U$,

$$S^*(\alpha) = \left\{ f \in A, \ \text{Re}\, \frac{zf'(z)}{f(z)} > \alpha, \ z \in U \right\}$$

the class of starlike functions of order $\alpha$, $0 \le \alpha < 1$,

$$M_\alpha = \left\{ f \in A, \ \text{Re}\, \left[ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf''(z)}{f(z)} + 1 \right) \right] > 0, \ z \in U \right\}$$

the class of $\alpha$-convex functions and

$$K = \left\{ f \in A : \ \text{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}$$

the class of convex functions.

**Theorem 1.** *[1] If $\alpha \in \mathbb{R}$ and $f \in M_\alpha$, then $f$ is a starlike function.*

**Theorem 2.** *[1] If $\alpha \geq 0$ and $f \in A$, then $f \in M_\alpha$ if and only if*

$$F(z) = f(z)\left[\frac{zf'(z)}{f(z)}\right]^\alpha$$

*is starlike.*

Let $\alpha \in \mathbb{C}$ and $f \in A$. We say that the function $f$ is $\alpha$-starlike if the function $F : U \to \mathbb{C}$, where

$$F(z) = f(z)\left[1 - \alpha + \alpha\frac{zf'(z)}{f(z)}\right], \quad z \in U$$

is starlike [4].

**Lemma 1.** *[1] Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ satisfy the condition*

$$\text{Re } \psi(is, t) \leq 0,$$

*for $s \in \mathbb{R}$, $t \leq -\frac{n}{2}(1 + s^2)$.*

*If $p(z) = 1 + p_n z^n + \ldots$ satisfies*

$$\text{Re } \psi[p(z), zp'(z)] > 0,$$

*then*

$$\text{Re } p(z) > 0.$$

## 2 Main results

**Definition 1.** *Let $\alpha$, $\beta \in \mathbb{R}$ and let $f \in A_n$, with*

$$\frac{f(z)f'(z)}{z} \neq 0, \quad 1 - \alpha + \alpha\frac{zf'(z)}{f(z)} \neq 0, \quad z \in U.$$

*We say that the function $f$ belongs to the class $M^n_{\alpha,\beta}$ if the function $F : U \to \mathbb{C}$ defined by*

$$F(z) = f(z)\left[\frac{zf'(z)}{f(z)}\right]^{\alpha(1-\beta)}\left[1 - \alpha + \alpha\frac{zf'(z)}{f(z)}\right]^\beta \tag{1}$$

*is starlike function.*

**Theorem 3.** *For any real number $\alpha$ and $\beta$ satisfying the condition $\alpha\beta(1-\alpha) \geq 0$ we have the inclusion $M^n_{\alpha,\beta} \subset S^*$.*

Proof: If we let

$$\frac{zf'(z)}{f(z)} = p(z), \quad z \in U,$$

then from the condition

$$\frac{f(z)f'(z)}{z} \neq 0, \quad z \in U$$

we deduce that the function $p$ is holomorphic in $U$.

Since $F$ is starlike we have

$$\operatorname{Re} \frac{zF'(z)}{F(z)} > 0, \quad z \in U \tag{2}$$

and from (1) we deduce

$$\frac{zF'(z)}{F(z)} = p(z) + \alpha(1-\beta)\frac{zp'(z)}{p(z)} + \alpha\beta\frac{zp'(z)}{1-\alpha+\alpha p(z)} = \psi[p(z), zp'(z)]. \tag{3}$$

Hence condition (2) is equivalent to

$$\operatorname{Re} \psi[p(z), zp'(z)] > 0, \quad z \in U. \tag{4}$$

Since

$$\operatorname{Re} (is, t) = \operatorname{Re} \left[is + \alpha(1-\beta)\frac{t}{is} + \alpha\beta\frac{t}{1-\alpha+\alpha p(z)}\right] =$$
$$= \alpha\beta t(1-\alpha)\frac{1}{(1-\alpha)^2+\alpha^2 s^2} \leq 0,$$

for all real $s$ and $t \leq \frac{-n}{2}(1+s^2)$, by using the Lemma we deduce that $\operatorname{Re} p(z) > 0$, which shows that $M^n_{\alpha,\beta} \subset S^*$. □

**Remarks.** 1) If $\beta = 0$ then the function $F : U \to \mathbb{C}$ is given by

$$F(z) = f(z)\left[\frac{zf'(z)}{f(z)}\right]^{\alpha}$$

and the class of functions $M^n_{\alpha,\beta} = M_\alpha$, is the class of $\alpha$-convex functions defined by P.T. Mocanu [2]. 2) If $\beta = 1$, then the function $F : U \to \mathbb{C}$ is given by

$$F(z) = f(z)\left[1 - \alpha + \alpha\frac{zf'(z)}{f(z)}\right]$$

and $M^n_{\alpha,1}$ is the class of $\alpha$-starlike functions defined by N.N. Pascu [4]. 3) If $\alpha = 0$, then the function $F : U \to \mathbb{C}$ is given by $F(z) = f(z)$ and $M^n_{0,\beta} = S^*$. 4) If $\alpha = 1$, then $F : U \to \mathbb{C}$ is given by $F(z) = zf'(z)$ and $M^n_{1,\beta} = K$.

**Definition 2.** *Let $\alpha$, $\beta \in \mathbb{R}$ and let $f \in A_n$ with*

$$\frac{f(z)f'(z)}{z} \neq 0, \quad 1-\alpha+\alpha\frac{zf'(z)}{f(z)} \neq 0, \quad z \in U.$$

*We say that the function $f$ belongs to the class $M^n_{\alpha,\beta}[\delta(\alpha,\beta,n)]$ if the function $F : U \to \mathbb{C}$ defined by (1) is starlike of order $\delta(\alpha,\beta,n)$, where*

$$\delta(\alpha,\beta,n) = \begin{cases} -\dfrac{\alpha\beta n}{2(1-\alpha)} & \text{if } 0 \leq \alpha \leq \dfrac{1}{2} \\[2ex] -\dfrac{(1-\alpha)\beta n}{2\alpha} & \text{if } \dfrac{1}{2} \leq \alpha \leq 1 \end{cases} \tag{5}$$

**Theorem 4.** *For any real number $\alpha$ and $\beta$ satisfying the condition $\alpha\beta(1-\alpha) \geq 0$ we have the inclusion $M^n_{\alpha,\beta}[\delta(\alpha,\beta,b)] \subset S^*$.*

PROOF: If we let

$$\frac{zf'(z)}{f(z)} = p(z), \quad z \in U,$$

then from the condition

$$\frac{f(z)f'(z)}{z} \neq 0,$$

we deduce that the function $p$ is holomorphic in $U$.

Since $F$ satisfies

$$\text{Re}\ \frac{zF'(z)}{F(z)} > \delta(\alpha,\beta,n) \tag{6}$$

with $\delta(\alpha,\beta,n)$ given by (5), from (1) we deduce

$$\frac{zF'(z)}{F(z)} - \delta(\alpha,\beta,n) = p(z) + \alpha(1-\beta)\frac{zp'(z)}{p(z)} + \alpha\beta\frac{zp'(z)}{1-\alpha+\alpha p(z)} - \delta(\alpha,\beta,n) = \psi[p(z), zp'(z)].$$

The condition (6) is equivalent to

$$\text{Re}\ \psi[p(z), zp'(z)] > 0, \quad z \in U. \tag{7}$$

We have

$$\begin{aligned}\text{Re}\ \psi(is,t) &= \text{Re}\ \left[is + \alpha(1-\beta)\frac{t}{is} + \alpha\beta\frac{t}{1-\alpha+\alpha is}\right] - \delta(\alpha,\beta,n) = \\ &= \frac{\alpha\beta t(1-\alpha)}{(1-\alpha)^2+\alpha^2s^2} - \delta(\alpha,\beta,n) \leq -\frac{\alpha\beta n(1-\alpha)(1+s^2)}{2[(1-\alpha)^2+\alpha^2s^2]} - \delta(\alpha,\beta,n) = \\ &= -\left[\frac{\alpha\beta n(1-\alpha)(1+s^2)}{2[(1-\alpha)^2+\alpha^2s^2]} + \delta(\alpha,\beta,n)\right] \leq 0.\end{aligned}$$

Hence by the Lemma we deduce Re $p(z) > 0$, which shows that

$$M^n_{\alpha,\beta}[\delta(\alpha,\beta,n)] \subset S^*.$$

□

## References


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