LIBERTAS MATHEMATICA, VOL XXII (2002)

LIFTS OF THE ALMOST SYMPLECTIC STRUCTURES TO $T(Osc^2 M)$

Classroom Notes

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Abstract. The geometry of the k-osculator bundle $(Osc^k M, \pi, M)$ has been studied by R. Miron and Gh. Atanasiu in their joint papers [3-4]. Obviously, the osculator bundle of second order, or the bundle of accelerations correspond to the case $k = 2$, [2], [3].

In the present paper we consider the group $\mathcal{G}_{as}$ of transformations of almost symplectic N-linear connections on $Osc^2 M$ and we determine its invariants, which are d-tensor fields. By means of these 27 invariants, we get characterizations of the integrability for the almost symplectic d-structures on $Osc^2 M$.

Concerning the terminology and notations, we use those from [2], [5], which are essentially based on M. Matsumoto's book [1].

Key words and phrases: almost symplectic structure, bundle of accelerations, integrability, invariants.

AMS 2000 Classification code: 53C05.

1 Preliminaries

Let M be a real C^∞-manifold with n dimensions, $n = 2n'$ and let $(Osc^2 M, \pi, M)$ be its 2-osculator bundle.

Let N be a nonlinear connection on $Osc^2 M$ with the coefficients $N_{(1)}{}^i_j$, $N_{(2)}{}^i_j$, $(i, j = 1, 2, \dots, n)$. The local coordinates on the $3n$-dimensional, manifold $E = Osc^2 M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$. Hence, the tangent space of E in the point $u = (x, y^{(1)}, y^{(2)}) \in E$ is given by the direct sum of the vector spaces:

$$T_u E = N_0(u) \oplus N_1(u) \oplus V_2(u), \qquad (\forall) u \in E. \tag{1.1}$$

An adapted basis to (1.1) is given by:

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}} \right\}, \qquad (i = 1, 2, \dots, n), \tag{1.2}$$

where:

$$\begin{cases} \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - N_{(1)}{}^j{}_i \dfrac{\partial}{\partial y^{(1)j}} - N_{(2)}{}^j{}_i \dfrac{\partial}{\partial y^{(2)j}}, \\ \dfrac{\delta}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - N_{(1)}{}^j{}_i \dfrac{\partial}{\partial y^{(2)j}}, \ \dfrac{\delta}{\delta y^{(2)i}} = \dfrac{\partial}{\partial y^{(2)i}}. \end{cases} \tag{1.3}$$

Let us consider the dual basis of (1.2):

$$\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\},\ (i = 1, 2, ..., n), \qquad \text{where} \tag{1.4}$$

$$\begin{cases} \delta x^i = dx^i,\ \delta y^{(1)i} = dy^{(1)i} + N_{(1)}{}^i{}_j dx^j, \\ \delta y^{(2)i} = dy^{(2)i} + N_{(1)}{}^i{}_j dy^{(1)j} + (N_{(2)}{}^i{}_j + N_{(1)}{}^i{}_m N_{(1)}{}^m{}_j) dx^j. \end{cases} \tag{1.5}$$

Let D be an N-linear connection on E, with the coefficients in the adapted basis: $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ and let a_{ij} be an almost symplectic d-structure on E, i.e. a d-tensor field: $a_{ij}(x, y^{(1)}, y^{(2)})$ of type (0, 2) on E, alternate and non-degenerate.

Then the d-tensor fields:

$$\Phi^{ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j - a_{sj} a^{ir}), \Phi^{*ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j + a_{sj} a^{ir}),$$

are called the Obata operators of the d-structure a_{ij}.

Let D be an almost symplectic N-linear connection on E, i.e. the h-and v_α-covariant derivatives of a_{ij} vanish: $a_{ij|k} = 0$, $a_{ij} \overset{(\alpha)}{|}_k = 0$, $(\alpha = 1, 2)$.

2 The group of transformations of almost symplectic N-linear connections

Let us consider the transformations $D\Gamma(N) \longrightarrow D\bar{\Gamma}(N)$ of almost symplectic N-linear connections. Owing to the Theorem 3.3., [7], they are given by:

$$\bar{L}^i_{jk} = L^i_{jk} + \Phi^{ir}_{sj} X^s_{rk},\ \bar{C}_{(\alpha)}{}^i_{jk} = C_{(\alpha)}{}^i_{jk} + \Phi^{ir}_{sj} Y_{(\alpha)}{}^s_{rk},\ (\alpha = 1, 2), \tag{2.1}$$

where X^i_{jk}, $Y_{(\alpha)}{}^i_{jk}$, $(\alpha = 1, 2)$ are arbitrary given d-tensor fields.

We shall pay attention to the invariants of the group $\mathcal{G}_{as}$.

The torsion and curvature d-tensor fields $T_{(0)}{}^i{}_{jk}$, $S_{(\alpha)}{}^i{}_{jk}$, $R_{(0\alpha)}{}^i{}_{jk}$, $R_{(12)}{}^i{}_{jk}$, $P_{(\alpha\beta)}{}^i{}_{jk}$, $Q_{(\alpha\beta)}{}^i{}_{jk}$, $(\alpha = 1, 2;\ \beta = 1, 2)$, $R_h{}^i{}_{jk}$, $P_{(\alpha)h}{}^i{}_{jk}$, $S_{(21)h}{}^i{}_{jk}$, $S_{(\alpha\alpha)h}{}^i{}_{jk}$, $(\alpha = 1, 2)$ are given in §3.4.[2].

Theorem 2.1. *By a transformation of almost symplectic N-linear connections (2.1), the d-tensors of torsion and curvature are transformed according to the following laws:*

$$\begin{aligned}
&\bar{T}_{(0)}{}^i{}_{jk} = T_{(0)}{}^i{}_{jk} + \mathcal{A}_{jk}\{\Phi^{ir}_{sj} X^s_{rk}\}, \quad \bar{R}_{(0\alpha)}{}^i{}_{jk} = R_{(0\alpha)}{}^i{}_{jk},\ (\alpha = 1, 2), \\
&\bar{P}_{(\alpha\alpha)}{}^i{}_{jk} = P_{(\alpha\alpha)}{}^i{}_{jk} - \Phi^{ir}_{sk} X^s_{rj},\ (\alpha = 1, 2), \quad \bar{Q}_{(21)}{}^i{}_{jk} = Q_{(21)}{}^i{}_{jk}, \\
&\bar{P}_{(12)}{}^i{}_{jk} = P_{(12)}{}^i{}_{jk}, \bar{P}_{(21)}{}^i{}_{jk} = P_{(21)}{}^i{}_{jk}, \quad \bar{S}_{(\alpha)}{}^i{}_{jk} = S_{(\alpha)}{}^i{}_{jk} + \mathcal{A}_{jk}\{\Phi^{ir}_{sj} Y_{(\alpha)}{}^s_{rk}\}, \\
&\bar{Q}_{(12)}{}^i{}_{jk} = Q_{(12)}{}^i{}_{jk} + \Phi^{ir}_{sj} Y_{(2)}{}^s_{rk}, \quad \bar{Q}_{(22)}{}^i{}_{jk} = Q_{(22)}{}^i{}_{jk} - \Phi^{ij}_{sk} Y_{(1)}{}^s_{rj}, \\
&\bar{R}_h{}^i{}_{jk} = R_h{}^i{}_{jk} + \Phi^{ir}_{sh} X^s_{rm} T_{(0)}{}^m{}_{jk} + \Phi^{ir}_{sh} Y_{(1)}{}^s_{rm} R_{(01)}{}^m{}_{jk} + \Phi^{ir}_{sh} Y_{(2)}{}^s_{rm} R_{(02)}{}^m{}_{jk} + \\
&\qquad + \mathcal{A}_{jk}\{\Phi^{ir}_{mh} X^m_{ri|k} + \Phi^{sr}_{mh} X^m_{rj} \Phi^{il}_{ps} X^p_{lk}\}, \ \bar{R}_{(12)}{}^i{}_{jk} = R_{(12)}{}^i{}_{jj},
\end{aligned} \tag{2.2}$$

$$
\begin{aligned}
\bar{P}_{(\alpha)h}{}^{i}{}_{jk} &= P_{(\alpha)h}{}^{i}{}_{jk} + \Phi^{ir}_{sh} Y_{(1)}{}^{s}_{rm} P_{(\alpha 1)}{}^{m}{}_{jk} + \Phi^{ir}_{sh} Y_{(2)}{}^{s}_{rm} P_{(\alpha 2)}{}^{m}{}_{jk} + \Phi^{ir}_{sh} X^{s}_{rm} C_{(\alpha)}{}^{m}_{jk} + \\
&\quad + \Phi^{ir}_{sh} X^{s}_{rj} \overset{(\alpha)}{|}_{k} - \Phi^{ir}_{sh} Y_{(\alpha)}{}^{s}_{rk|j} + \Phi^{sr}_{mh} X^{m}_{rj} \Phi^{il}_{ps} Y_{(\alpha)}{}^{p}_{lk} - \Phi^{sr}_{mh} Y_{(\alpha)}{}^{m}_{rj} \Phi^{il}_{ps} X^{p}_{lk}, \\
\bar{S}_{(21)h}{}^{i}{}_{jk} &= S_{(21)h}{}^{i}{}_{jk} + \Phi^{ir}_{sh} Y_{(2)}{}^{s}_{rm} P_{(21)}{}^{m}{}_{jk} + \Phi^{ir}_{sh} Y_{(1)}{}^{s}_{rm} C_{(2)}{}^{m}_{jk} - \Phi^{ir}_{sh} Y_{(2)}{}^{s}_{rm} C_{(1)}{}^{m}_{kj} + \\
&\quad + \Phi^{ir}_{sh} Y_{(1)}{}^{s}_{rj} \overset{(2)}{|}_{k} - \Phi^{ir}_{sh} Y_{(2)}{}^{s}_{rk} \overset{(1)}{|}_{j} + \Phi^{sr}_{mh} Y_{(1)}{}^{m}_{rj} \Phi^{il}_{ps} Y_{(2)}{}^{p}_{lk} - \Phi^{sr}_{mh} Y_{(2)}{}^{m}_{rk} \Phi^{il}_{ps} Y_{(1)}{}^{p}_{lj}, \\
\bar{S}_{(\alpha\alpha)h}{}^{i}{}_{jk} &= S_{(\alpha\alpha)h}{}^{i}{}_{jk} + \Phi^{ir}_{sh} Y_{(\alpha)}{}^{s}_{rm} S_{(\alpha)}{}^{m}{}_{jk} + \Phi^{ir}_{sh} Y_{(\alpha)}{}^{s}_{rm} R_{(\alpha 2)}{}^{m}{}_{jk} + \\
&\quad + \mathcal{A}_{jk}\{\Phi^{ir}_{sh} Y_{(\alpha)}{}^{s}_{rj} \overset{(\alpha)}{|}_{k} + \Phi^{sr}_{mh} Y_{(\alpha)}{}^{m}_{rj} \Phi^{il}_{ps} Y_{(\alpha)}{}^{p}_{lk}\}, \ \alpha = 1,2, \ R_{(22)}{}^{m}{}_{jk} = 0.
\end{aligned}
$$

We make some notations:

$$
\begin{aligned}
t_{(12)}{}^{i}{}_{jk} &= \mathcal{A}_{jk}\{\tfrac{\delta N_{(2)}{}^{i}{}_{j}}{\delta y^{(1)k}}\}, \ t_{(2\alpha)}{}^{i}{}_{jk} = \mathcal{A}_{jk}\{\tfrac{\partial N_{(\alpha)}{}^{i}{}_{j}}{\partial y^{(2)k}}\}, \ (\alpha = 1,2), \\
t_{(2\alpha)}{}^{*}{}_{ijk} &= \mathcal{S}_{ijk}\{a_{im} t_{(2\alpha)}{}^{m}{}_{jk}\}, \ (\alpha = 1,2); \ t_{(12)}{}^{*}{}_{ijk} = \mathcal{S}_{ijk}\{a_{im} t_{(12)}{}^{m}{}_{jk}\}, \\
T_{(0)}{}^{*}{}_{ijk} &= \mathcal{S}_{ijk}\{a_{im} T_{(0)}{}^{m}{}_{jk}\}, \ S_{(\alpha)}{}^{*}{}_{ijk} = \mathcal{S}_{ijk}\{a_{im} S_{(\alpha)}{}^{m}{}_{jk}\}, \ (\alpha = 1,2), \\
Q_{(21)}{}^{*}{}_{ijk} &= \mathcal{S}_{ijk}\{a_{im} Q_{(21)}{}^{m}{}_{jk}\}, \ R_{(0\alpha)}{}^{*}{}_{ijk} = \mathcal{S}_{ijk}\{a_{im} R_{(0\alpha)}{}^{m}{}_{jk}\}, \\
R_{(12)}{}^{*}{}_{ijk} &= \mathcal{S}_{ijk}\{a_{im} R_{(12)}{}^{m}{}_{jk}\}, \ \overset{1}{S}_{ijk} = a_{mj} P_{(22)}{}^{m}{}_{ik} - a_{mk} P_{(11)}{}^{m}{}_{ij}, \\
\overset{1}{k}_{(\alpha\alpha)ijk} &= a_{km} T_{(0)}{}^{m}{}_{ij} + \mathcal{A}_{ij}\{a_{im} P_{(\alpha\alpha)}{}^{m}{}_{jk}\}, \ (\alpha = 1,2), \\
\overset{2}{k}_{(\alpha)ijk} &= a_{im} S_{(\alpha)}{}^{m}{}_{jk} + \mathcal{A}_{jk}\{a_{km} C_{(\alpha)}{}^{m}_{ij}\}, \ (\alpha = 1,2), \\
\overset{3}{k}_{(\alpha\beta)ijk} &= \mathcal{A}_{jk}\{a_{km} P_{(\alpha\beta)}{}^{m}{}_{ij}\}, \ (\alpha,\beta = 1,2), \\
\overset{4}{k}_{(\alpha)ijk} &= \mathcal{A}_{ij}\{a_{im} C_{(\alpha)}{}^{m}_{jk}\}, \ (\alpha = 1,2), \\
\overset{1}{\mathcal{N}}_{(22)ijk} &= a_{km} S_{(1)}{}^{m}{}_{ij} + \mathcal{A}_{ij}\{a_{im} Q_{(22)}{}^{m}{}_{jk}\}, \\
\overset{3}{\mathcal{N}}_{(22)ijk} &= \mathcal{A}_{jk}\{a_{km} Q_{(22)}{}^{m}{}_{ij}\}, \\
\overset{2}{S}_{(\alpha\beta)ijk} &= \mathcal{A}_{jk}\{a_{km} P_{(\alpha\beta)}{}^{m}{}_{ij}\}, \ (\alpha,\beta = 1,2, \ \alpha \neq \beta),
\end{aligned}
\tag{2.3}
$$

where $\mathcal{A}_{jk}\{...\}$ denotes the alternate summation: $\mathcal{A}_{jk}\{B_{jk}\} = B_{jk} - B_{kj}$ and $\mathcal{S}_{ijk}\{...\}$ denotes the cyclic summation: $\mathcal{S}_{ijk}\{B_{ijk}\} = B_{ijk} + B_{jki} + B_{kij}$.
By direct calculation from (2.1) and (2.2), we have:

Theorem 2.2. *The d-tensor fields* $R_{(0\alpha)}{}^{i}{}_{jk}$, $R_{(12)}{}^{i}{}_{jk}$, $t_{(12)}{}^{i}{}_{jk}$, $t_{(2\alpha)}{}^{i}{}_{jk}$, $Q_{(21)}{}^{i}{}_{jk}$, $P_{(12)}{}^{i}{}_{jk}$, $P_{(21)}{}^{i}{}_{jk}$, $t_{(12)}{}^{*}{}_{ijk}$, $t_{(2\alpha)}{}^{*}{}_{ijk}$, $T_{(0)}{}^{*}{}_{ijk}$, $S_{(\alpha)}{}^{*}{}_{ijk}$, $Q_{(21)}{}^{*}{}_{ijk}$, $R_{(0\alpha)}{}^{*}{}_{ijk}$, $\overset{1}{k}_{(\alpha\alpha)ijk}$, $\overset{2}{k}_{(\alpha)ijk}$, $\overset{3}{k}_{(\alpha\beta)ijk}$, $\overset{4}{k}_{(\alpha)ijk}$, $\overset{1}{\mathcal{N}}_{(22)ijk}$, $\overset{3}{\mathcal{N}}_{(22)ijk}$, $\overset{2}{S}_{(\alpha\beta)ijk}$, $\overset{1}{S}_{ijk}$, $(\alpha = 1,2; \ \beta = 1,2)$, are *invariants of the group* $\mathcal{G}_{as}$.

3 2-forms on $T(Osc^2M)$

Let $\Lambda^k(T(Osc^2M))$ be the $\mathcal{F}$-module of all k-forms on the tangent bundle $T(Osc^2M)$, where $\mathcal{F}(T(Osc^2M))$ is the ring of all differentiable functions on $T(Osc^2M)$. If N is a given nonlinear connection, then $\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\}$ makes a local basis of $\Lambda^1(T(Osc^2M))$, which is dual to $\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}}\right\}$.

If $f \in \mathcal{F}(Osc^2M)$, then the 1-form df is written as:

$$df = \frac{\delta f}{\delta x^i}dx^i + \frac{\delta f}{\delta y^{(1)i}}\delta y^{(1)i} + \frac{\partial f}{\partial y^{(2)i}}\delta y^{(2)i} \tag{3.1}$$

and the exterior differentials of $dx^i, \delta y^{(1)i}, \delta y^{(2)i}$ are given by:

$$\begin{cases} d(dx^i) = 0, \\ d(\delta y^{(1)i}) = \frac{1}{2}R_{(01)}{}^i{}_{jm}dx^m \wedge dx^j + B_{(11)}{}^i{}_{jm}\delta y^{(1)m} \wedge dx^j + \\ \quad +B_{(21)}{}^i{}_{jm}\delta y^{(2)m} \wedge dx^j, \\ d(\delta y^{(2)i}) = \frac{1}{2}R_{(02)}{}^i{}_{jm}dx^m \wedge dx^j + B_{(12)}{}^i{}_{jm}\delta y^{(1)m} \wedge dx^j + \\ \quad +B_{(22)}{}^i{}_{jm}\delta y^{(2)m} \wedge dx^j + \frac{1}{2}R_{(12)}{}^i{}_{jm}\delta y^{(1)m} \wedge \delta y^{(1)j} + \\ \quad +B_{(21)}{}^i{}_{jm}\delta y^{(2)m} \wedge \delta y^{(1)j}. \end{cases} \tag{3.2}$$

Generally, $\omega \in \Lambda^2(T(Osc^2M))$ is written in the form:

$$\begin{aligned} \omega = &\frac{1}{2}\tilde{a}_{ij}dx^i \wedge dx^j + \tilde{b}_{ij}dx^i \wedge \delta y^{(1)j} + \tilde{c}_{ij}dx^i \wedge \delta y^{(2)j} + \\ &+\frac{1}{2}\tilde{d}_{ij}\delta y^{(1)i} \wedge \delta y^{(1)j} + \tilde{e}_{ij}\delta y^{(1)i} \wedge \delta y^{(2)j} + \frac{1}{2}\tilde{f}_{ij}\delta y^{(2)i} \wedge \delta y^{(2)j}, \end{aligned} \tag{3.3}$$

where $\tilde{a}_{ij} = -\tilde{a}_{ji},\ \tilde{d}_{ij} = -\tilde{d}_{ji},\ \tilde{f}_{ij} = -\tilde{f}_{ji}$.

The exterior differential $d\omega$ is given by:

$$\begin{aligned} d\omega = &\frac{1}{6}\overset{1}{\omega}_{ijk}\, dx^i \wedge dx^j \wedge dx^k + \frac{1}{2}\overset{2}{\omega}_{ijk}\, dx^i \wedge dx^j \wedge \delta y^{(1)k} + \\ &+\frac{1}{2}\overset{3}{\omega}_{ijk}\, dx^i \wedge dx^j \wedge \delta y^{(2)k} + \frac{1}{2}\overset{4}{\omega}_{ijk}\, dx^i \wedge \delta y^{(1)j} \wedge \delta y^{(1)k} + \\ &+\overset{5}{\omega}_{ijk}\, dx^i \wedge \delta y^{(1)j} \wedge \delta y^{(2)k} + \frac{1}{2}\overset{6}{\omega}_{ijk}\, dx^i \wedge \delta y^{(2)j} \wedge \delta y^{(2)k} + \\ &+\frac{1}{6}\overset{7}{\omega}_{ijk}\, \delta y^{(1)i} \wedge \delta y^{(1)j} \wedge \delta y^{(1)k} + \frac{1}{2}\overset{8}{\omega}_{ijk}\, \delta y^{(1)i} \wedge \delta y^{(1)j} \wedge \delta y^{(2)k} + \\ &+\frac{1}{2}\overset{9}{\omega}_{ijk}\, \delta y^{(1)i} \wedge \delta y^{(2)j} \wedge \delta y^{(2)k} + \frac{1}{6}\overset{10}{\omega}_{ijk}\, \delta y^{(2)i} \wedge \delta y^{(2)j} \wedge \delta y^{(2)k}, \end{aligned} \tag{3.4}$$

where:

$$\overset{1}{\omega}_{ijk} = S_{ijk}\{\frac{\delta \tilde{a}_{ij}}{\delta x^k} + \tilde{b}_{im}R_{(01)}{}^m{}_{jk} + \tilde{c}_{im}R_{(02)}{}^m{}_{jk}\},$$

$$
\begin{aligned}
\overset{2}{\omega}_{ijk} &= \frac{\delta \tilde{a}_{ij}}{\delta y^{(1)k}} + \tilde{d}_{km} R_{(01)}{}^{m}{}_{ij} + \tilde{e}_{km} R_{(02)}{}^{m}{}_{ij} + \mathcal{A}_{ij}\{\frac{\delta \tilde{b}_{jk}}{\delta x^i} + \tilde{b}_{im} B_{(11)}{}^{m}{}_{jk} + \tilde{c}_{im} B_{(12)}{}^{m}{}_{jk}\},\\
\overset{3}{\omega}_{ijk} &= \frac{\partial \tilde{a}_{ij}}{\partial y^{(2)k}} - \tilde{e}_{mk} R_{(01)}{}^{m}{}_{ij} + \tilde{f}_{km} R_{(02)}{}^{m}{}_{ij} + \mathcal{A}_{ij}\{\frac{\delta \tilde{c}_{jk}}{\delta x^i} + \tilde{b}_{im} B_{(21)}{}^{m}{}_{jk} + \tilde{c}_{im} B_{(22)}{}^{m}{}_{jk}\},\\
\overset{4}{\omega}_{ijk} &= \frac{\delta \tilde{d}_{jk}}{\delta x^i} + \tilde{c}_{im} R_{(12)}{}^{m}{}_{jk} + \mathcal{A}_{jk}\{\frac{\delta \tilde{b}_{ij}}{\delta y^{(1)k}} + \tilde{d}_{km} B_{(11)}{}^{m}{}_{ij} + \tilde{e}_{km} B_{(12)}{}^{m}{}_{ij}\},\\
\overset{5}{\omega}_{ijk} &= \frac{\partial \tilde{b}_{ij}}{\partial y^{(2)k}} - \frac{\delta \tilde{c}_{ik}}{\delta y^{(1)j}} + \frac{\delta \tilde{e}_{jk}}{\delta x^i} + \tilde{c}_{im} B_{(21)}{}^{m}{}_{jk} - \tilde{d}_{jm} B_{(21)}{}^{m}{}_{ik} - \tilde{e}_{jm} B_{(22)}{}^{m}{}_{ik} -\\
&\quad - \tilde{e}_{mk} B_{(11)}{}^{m}{}_{ij} + \tilde{f}_{km} B_{(12)}{}^{m}{}_{ij}, \qquad (3.5)\\
\overset{6}{\omega}_{ijk} &= \frac{\delta \tilde{f}_{jk}}{\delta x^i} + \mathcal{A}_{jk}\{\frac{\partial \tilde{c}_{ij}}{\partial y^{(2)k}} - \tilde{e}_{mk} B_{(21)}{}^{m}{}_{ij} + \tilde{f}_{km} B_{(22)}{}^{m}{}_{ij}\},\\
\overset{7}{\omega}_{ijk} &= \mathcal{S}_{ijk}\{\frac{\delta \tilde{d}_{ij}}{\delta y^{(1)k}} + \tilde{e}_{im} R_{(12)}{}^{m}{}_{jk}\},\\
\overset{8}{\omega}_{ijk} &= \frac{\partial \tilde{d}_{ij}}{\partial y^{(2)k}} + \tilde{f}_{km} R_{(12)}{}^{m}{}_{ij} + \mathcal{A}_{ij}\{\frac{\delta \tilde{e}_{jk}}{\delta y^{(1)i}} + \tilde{e}_{im} B_{(21)}{}^{m}{}_{jk}\},\\
\overset{9}{\omega}_{ijk} &= \frac{\delta \tilde{f}_{jk}}{\delta y^{(1)i}} + \mathcal{A}_{jk}\{\frac{\partial \tilde{e}_{ij}}{\partial y^{(2)k}} + \tilde{f}_{km} B_{(21)}{}^{m}{}_{ij}\},\ \overset{10}{\omega}_{ijk} = \mathcal{S}_{ijk}\{\frac{\partial \tilde{f}_{ij}}{\partial y^{(2)k}}\}.
\end{aligned}
$$

Proposition 3.1. *If an N-linear connection D is given, with the local coefficients: $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk})$, then $\overset{\alpha}{\omega}_{ijk}$, $(\alpha = 1, 2, ..., 10)$ have the following expressions:*

$$
\begin{aligned}
\overset{1}{\omega}_{ijk} &= \mathcal{S}_{ijk}\{\tilde{a}_{ij|k} + \tilde{a}_{im} T_{(0)}{}^{m}{}_{jk} + \tilde{b}_{im} R_{(01)}{}^{m}{}_{jk} + \tilde{c}_{im} R_{(02)}{}^{m}{}_{jk}\},\\
\overset{2}{\omega}_{ijk} &= \tilde{a}_{ij} \overset{(1)}{|}_k + \tilde{b}_{mk} T_{(0)}{}^{m}{}_{ji} + \tilde{d}_{km} R_{(01)}{}^{m}{}_{ij} + \tilde{e}_{km} R_{(02)}{}^{m}{}_{ij} + \mathcal{A}_{ij}\{\tilde{b}_{jk|i} + \tilde{a}_{im} C_{(1)}{}^{m}_{jk} +\\
&\quad + \tilde{b}_{im} P_{(11)}{}^{m}{}_{jk} + \tilde{c}_{im} P_{(12)}{}^{m}{}_{jk}\},\\
\overset{3}{\omega}_{ijk} &= \tilde{a}_{ij} \overset{(2)}{|}_k + \tilde{c}_{mk} T_{(0)}{}^{m}{}_{ji} + \tilde{e}_{mk} R_{(01)}{}^{m}{}_{ji} + \tilde{f}_{km} R_{(02)}{}^{m}{}_{ij} + \mathcal{A}_{ij}\{\tilde{c}_{jk|i} + \tilde{a}_{im} C_{(2)}{}^{m}_{jk} +\\
&\quad + \tilde{b}_{im} P_{(21)}{}^{m}{}_{jk} + \tilde{c}_{im} P_{(22)}{}^{m}{}_{jk}\},\\
\overset{4}{\omega}_{ijk} &= \tilde{d}_{jk|i} + \tilde{b}_{im} S_{(1)}{}^{m}{}_{jk} + \tilde{c}_{im} R_{(12)}{}^{m}{}_{jk} + \mathcal{A}_{jk}\{\tilde{b}_{ij} \overset{(1)}{|}_k + \tilde{b}_{mj} C_{(1)}{}^{m}_{ik} +\\
&\quad + \tilde{d}_{mj} P_{(11)}{}^{m}{}_{ik} - \tilde{e}_{jm} P_{(12)}{}^{m}{}_{ik}\},\\
\overset{5}{\omega}_{ijk} &= \tilde{b}_{ij} \overset{(2)}{|}_k - \tilde{c}_{ik} \overset{(1)}{|}_j + \tilde{e}_{jk|i} + \tilde{b}_{mj} C_{(2)}{}^{m}_{ik} + \tilde{b}_{im} C_{(2)}{}^{m}_{jk} + \tilde{c}_{im} Q_{(22)}{}^{m}{}_{jk} -\\
&\quad - \tilde{d}_{jm} P_{(21)}{}^{m}{}_{ik} - \tilde{e}_{jm} P_{(22)}{}^{m}{}_{ik} - \tilde{c}_{mk} C_{(1)}{}^{m}_{ij} + \tilde{f}_{km} P_{(12)}{}^{m}{}_{ij} - \tilde{e}_{mk} P_{(11)}{}^{m}{}_{ij}, \qquad (3.6)\\
\overset{6}{\omega}_{ijk} &= \tilde{f}_{jk|i} + \tilde{c}_{im} S_{(2)}{}^{m}{}_{jk} + \mathcal{A}_{jk}\{\tilde{c}_{ij} \overset{(2)}{|}_k + \tilde{c}_{mj} C_{(2)}{}^{m}_{ik} + \tilde{e}_{mj} P_{(21)}{}^{m}{}_{ik} + \tilde{f}_{mj} P_{(22)}{}^{m}{}_{ik}\},\\
\overset{7}{\omega}_{ijk} &= \mathcal{S}_{ijk}\{\tilde{d}_{ij} \overset{(1)}{|}_k + \tilde{d}_{im} S_{(1)}{}^{m}{}_{ik} + \tilde{e}_{im} R_{(12)}{}^{m}{}_{jk}\},\\
\overset{8}{\omega}_{ijk} &= \tilde{d}_{ij} \overset{(2)}{|}_k + \tilde{e}_{mk} S_{(1)}{}^{m}{}_{ji} + \tilde{f}_{km} R_{(12)}{}^{m}{}_{ij} + \mathcal{A}_{ij}\{\tilde{e}_{jk} \overset{(1)}{|}_i + \tilde{d}_{im} C_{(2)}{}^{m}_{jk} + \tilde{e}_{im} Q_{(22)}{}^{m}{}_{jk}\},
\end{aligned}
$$

$$\overset{9}{\omega}_{ijk} = \tilde{f}_{jk} \overset{(1)}{|}_i + \tilde{e}_{im} S_{(2)}{}^m{}_{jk} + \mathcal{A}_{jk}\{\tilde{e}_{ij} \overset{(2)}{|}_k + \tilde{e}_{mj} C_{(2)}{}^m_{ik} + \tilde{f}_{mj} Q_{(22)}{}^m{}_{ik}\},$$

$$\overset{10}{\omega}_{ijk} = \mathcal{S}_{ijk}\{\tilde{f}_{ij} \overset{(2)}{|}_k + \tilde{f}_{im} S_{(2)}{}^m{}_{jk}\}.$$

For $\omega \in \Lambda^2(T(Osc^2M))$ written in the form (3.6) we put:

$$A = \begin{pmatrix} \tilde{a}_{ij} & \tilde{b}_{ij} & \tilde{c}_{ij} \\ -\tilde{b}_{ji} & \tilde{d}_{ij} & \tilde{e}_{ij} \\ -\tilde{c}_{ji} & -\tilde{e}_{ji} & \tilde{f}_{ij} \end{pmatrix} \tag{3.7}$$

Definition 3.2. *A 2-form $\omega \in \Lambda^2(T(Osc^2M))$, for which the matrix A is nondegenerate, is called integrable if: $d\omega=0$.*

Theorem 3.3. *A 2-form $\omega \in \Lambda^2(T(Osc^2M))$, for which the matrix A is non-degenerate, is integrable if and only if the tensor fields $\overset{\alpha}{\omega}_{ijk}$, $(\alpha = 1, 2, ..., 10)$ vanish.*

Observation 3.1. A 2-form $\omega \in \Lambda^2(T(Osc^2M))$ with $detA \neq 0$ is called non-degenerate and determines an almost symplectic structure on $T(Osc^2M)$.

When $\tilde{b}_{ij} = 0$, $\tilde{c}_{ij} = 0$, $\tilde{e}_{ij} = 0$, then $\tilde{a}_{ij}$, $\tilde{d}_{ij}$, $\tilde{f}_{ij}$ give three almost symplectic structures on Osc^2M.

When $\tilde{a}_{ij} = 0$ or $\tilde{d}_{ij} = 0$ or $\tilde{f}_{ij} = 0$ and $\tilde{b}_{ij} = -\tilde{b}_{ji}$, $\tilde{c}_{ij} = -\tilde{c}_{ji}$, $\tilde{e}_{ij} = -\tilde{e}_{ji}$, then $\tilde{b}_{ij}$, $\tilde{c}_{ij}$, $\tilde{e}_{ij}$ give almost symplectic structures on Osc^2M.

Conversely, if a_{ij} is a given almost symplectic d-structure on Osc^2M, then the 2-forms on $T(Osc^2M)$ defined by: $\omega = \frac{1}{2}a_{ij}dx^i \wedge dx^j + \frac{1}{2}a_{ij}\delta y^{(1)i} \wedge \delta y^{(1)j} + \frac{1}{2}a_{ij}\delta y^{(2)i} \wedge \delta y^{(2)j}$, $\omega = a_{ij}dx^i \wedge \delta y^{(1)i} + a_{ij}dx^i \wedge \delta y^{(2)j} + a_{ij}\delta y^{(1)i} \wedge \delta y^{(2)j}$, etc. determine almost symplectic d-structures on $T(Osc^2M)$. The integrability of each of these 2-forms gives some types of integrability for the given almost symplectic d-structure a_{ij}.

4 The integrability of an almost symplectic d-structure in the bundle of accelerations

Assume that a nonlinear connection N is given, then an almost symplectic d-structure a_{ij} on the base manifold Osc^2M is lifted to a 2-form ω on $T(Osc^2M)$ in various ways. We consider the following ω of six single types: I, II, III, IV, V, VI and combined types (4.1), where $\alpha \neq \pm 1$, $\alpha \in R$, $\beta \neq 1$, $\beta \in R$:

$$\text{I: } \omega = \frac{1}{2}a_{ij}dx^i \wedge dx^j; \quad \text{II: } \omega = a_{ij}dx^i \wedge \delta y^{(1)j}; \quad \text{III: } \omega = a_{ij}dx^i \wedge \delta y^{(2)j}; \tag{4.1}$$
$$\text{IV: } \omega = \frac{1}{2}a_{ij}\delta y^{(1)i} \wedge \delta y^{(1)j}; \quad \text{V: } \omega = a_{ij}\delta y^{(1)i} \wedge \delta y^{(2)j}; \quad \text{VI: } \omega = \frac{1}{2}a_{ij}\delta y^{(2)i} \wedge \delta y^{(2)j},$$

and the combined types: I+V, II+VI, III+IV, I+II+V, I+II+VI, I+III+IV, I+III+V, I+IV+V, I+IV+VI, I+V+VI, II+III+IV, II+III+V, II+III+VI, II+ +IV+VI, II+V+VI, III+IV+V, III+IV+VI, I+II+III+IV, I+II+III+V, I+II+ +III+VI, I+II+IV+V, I+αII+IV+VI, I+III+IV+V, I+αIII+IV+VI, I+III+ +V+VI, I+II+V+VI, I+IV+αV+VI, II+III+IV+V, II+III+IV+VI, II+III+ +V+VI, II+IV+V+VI, III+IV+V+VI, I+II+III+IV+βV, I+II+III+IV+VI, I+βII+III+V+VI, I+II+IV+V+VI, I+III+IV+V+VI, βII+III+IV+V+VI, I+ +βII+III+IV+V+VI.

Proposition 4.1. *Each 2-form ω of the types (4.1) is non-degenerate and defines an almost symplectic structure on $T(Osc^2M)$.*

PROOF: I+V. We obtain $\omega = \frac{1}{2}a_{ij}dx^i \wedge dx^j + a_{ij}\delta y^{(1)i} \wedge \delta y^{(2)j}$. It follows that $A = \begin{pmatrix} a_{ij} & 0 & 0 \\ 0 & 0 & a_{ij} \\ 0 & a_{ji} & 0 \end{pmatrix}$ with $det A = \det\ (a_{ij})^3 \neq 0$, etc. □

Proposition 4.2. *The coefficients $\overset{\alpha}{\omega}_{ijk}$, $(\alpha = 1,2,...,10)$ of the exterior differentials of the 2-forms ω given in Proposition 4.1. are invariants of the group $\mathcal{G}_{as}$.*

PROOF: Calculating directly from Proposition 3.1. and using the notations (2.3) we have:

I: $\overset{1}{\omega}_{ijk} = T_{(0)}{}^{*}{}_{ijk}$, $\overset{2}{\omega}_{ijk} = \overset{4}{k}_{(1)ijk}$, $\overset{3}{\omega}_{ijk} = \overset{4}{k}_{(2)ijk}$;

II: $\overset{1}{\omega}_{ijk} = R_{(01)}{}^{*}{}_{ijk}$, $\overset{2}{\omega}_{ijk} = \overset{1}{k}_{(11)ijk}$, $\overset{3}{\omega}_{ijk} = \overset{2}{S}_{(21)ijk}$, $\overset{4}{\omega}_{ijk} = \overset{2}{k}_{(1)ijk}$, $\overset{5}{\omega}_{ijk} = \overset{4}{k}_{(2)ijk}$;

III: $\overset{1}{\omega}_{ijk} = R_{(02)}{}^{*}{}_{ijk}$, $\overset{2}{\omega}_{ijk} = \overset{2}{S}_{(12)ijk}$, $\overset{3}{\omega}_{ijk} = \overset{1}{k}_{(22)ijk}$, $\overset{4}{\omega}_{ijk} = a_{im}R_{(12)}{}^{m}{}_{jk}$,
$\overset{5}{\omega}_{ijk} = a_{im}P_{(21)}{}^{m}{}_{jk} + \overset{4}{k}_{(1)ijk}$, $\overset{6}{\omega}_{ijk} = \overset{2}{k}_{(2)ijk}$,

IV: $\overset{2}{\omega}_{ijk} = a_{km}R_{(01)}{}^{m}{}_{ij}$, $\overset{4}{\omega}_{ijk} = \overset{3}{k}_{(11)ijk}$, $\overset{5}{\omega}_{ijk} = a_{mj}P_{(21)}{}^{m}{}_{ik}$, $\overset{7}{\omega}_{ijk} = S_{(1)}{}^{*}{}_{ijk}$,
$\overset{8}{\omega}_{ijk} = \overset{4}{k}_{(2)ijk}$;

V: $\overset{2}{\omega}_{ijk} = a_{km}R_{(02)}{}^{m}{}_{ij}$, $\overset{3}{\omega}_{ijk} = a_{km}R_{(01)}{}^{m}{}_{ij}$, $\overset{4}{\omega}_{ijk} = \overset{3}{k}_{(12)ijk}$, $\overset{5}{\omega}_{ijk} = \overset{1}{S}_{ijk}$,
$\overset{6}{\omega}_{ijk} = \overset{3}{k}_{(21)ijk}$, $\overset{7}{\omega}_{ijk} = R_{(12)}{}^{*}{}_{ijk}$, $\overset{8}{\omega}_{ijk} = \overset{1}{\mathcal{N}}_{(22)ijk}$, $\overset{9}{\omega}_{ijk} = \overset{2}{k}_{(2)ijk}$;

VI: $\overset{3}{\omega}_{ijk} = a_{km}R_{(02)}{}^{m}{}_{ij}$, $\overset{5}{\omega}_{ijk} = a_{km}P_{(12)}{}^{m}{}_{ij}$, $\overset{6}{\omega}_{ijk} = \overset{3}{k}_{(22)ijk}$, $\overset{8}{\omega}_{ijk} = a_{km}R_{(12)}{}^{m}{}_{ij}$,
$\overset{9}{\omega}_{ijk} = \overset{3}{\mathcal{N}}_{(22)ijk}$, $\overset{10}{\omega}_{ijk} = S_{(2)}{}^{*}{}_{ijk}$.

The missing $\overset{\alpha}{\omega}$ vanish.

Since $\overset{\alpha}{\omega}_{ijk}$, $(\alpha = 1,2,...,10)$ are linear combinations of $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij}, \tilde{d}_{ij}, \tilde{e}_{ij}, \tilde{f}_{ij}$, the expressions for the combined types (4.1) are obtained as the linear combinations of the ones for: I,II,III,IV,V,VI. □

Corresponding to Definition 3.1. we have:

Definition 4.3. *An almost symplectic d-structure a_{ij} on a differentiable manifold Osc^2M is called integrable of the types given in (4.1), with respect to the nonlinear connection N if the corresponding lifted 2-forms on $T(Osc^2M)$ are integrable.*

Theorem 4.4. *An almost symplectic d-structure $a_{ij}(x, y^{(1)}, y^{(2)})$ on a differentiable manifold Osc^2M is integrable of the types given in (4.1), if and only if the invariants of the group $\mathcal{G}_{as}$, vanish.*

Observation 4.1. We note that the above integrabilities are reduced to four types: I+V, II+VI, III+IV and I+IV+VI, because the other types appear like linear combinations of null invariants of the group $\mathcal{G}_{as}$.

References

1. Matsumoto M., *The theory of Finsler connections*, Publ.Study Group Geom. 5, Depart.Math., Okayama Univ., 1970.
2. Miron R., *The Geometry of Higher-order Lagrange spaces. Applications in Mechanics and Physics*, Kluwer Academic Publishers, FTPH82, 1997.
3. Miron R. and Atanasiu Gh.,*Lagrange Geometry of Second Order*, Math. Comput. Modelling,Pergamon, 20, 4/5(1994), 41–56.
4. Miron R. and Atanasiu Gh.,*Differential Geometry of the k-osculator bundle*, Rev.Roumaine Math.Pures Appl., 41, 3/4(1996), 205–236.
5. Miron R. and Hashiguchi M.,*Almost Symplectic Finsler Structures*, Fac.Sci., Kagoshima Univ.(Math., Phys.& Chem.), 14(1981), 9–19.
6. Purcaru M. and Blănuţă V.,*The Integrability of Almost $2-\pi$ Structures in the Bundle of Accelerations*, An. Şt. Univ. "A.I.Cuza" Iaşi, XLII, Supl.Mat.(1996), 197–206.
7. Purcaru M.,*Almost Symplectic N-Linear Connections in the Bundle of Accelerations*, Novi Sad J.Math., Vol. 29, No. 3(1999), 281–289.