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The Homogeneous Prolongation to the Second Order Tangent Bundle T^2M of a Riemannian Metric

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Abstract. In some previous papers [2] [8] [10] the prolongation G given by (1.7) of a Riemannian metric g on a manifold M to the second order tangent bundle T^2M was considered. The terms appearing in the metric G, called also a Sasaki lift of g, have no the same homogeneity degree with respect to the natural homotheties on T^2M and so G is not homogeneous. This fact constitutes an inconvenient when the study is concentrated on the homogeneous objects on T^2M as well as when T^2M is taken as a model in a physical theory since the said terms have no the same physical dimensions. In this paper we show that such disadvantages can be overcome by replacing G with a modified metric $\overset{0}{G}$ given by (2.1) which is also of Sasaki type. All terms in $\overset{0}{G}$ are homogeneous of degree zero. Some properties of the Riemannian space $(\widetilde{T^2M}, \overset{0}{G})$ are pointed out. An $f(3,1)-$structure $\overset{0}{F}$ is introduced and it is proved that the pair $(\overset{0}{F}, \overset{0}{G})$ is a metrical $f(3,1)-$structure. The Nijenhuis tensor field of the structure $\overset{0}{F}$ vanishes iff the Riemannian metric g is of constant curvature.

Keywords: second order tangent bundle, Riemannian metrics, homogeneous prolongation.

Introduction

An old problem in differential geometry is that of the prolongation of the Riemannian structures defined on the manifold M to the higher order osculator bundle Osc^kM identified with the tangent bundle of order k, T^kM. This problem belongs to the so-called higher order geometry.

Several remarkable geometers as E. Bompiani, Ch. Ehresmann, S. Kobayashi have studied this problem. It has been solved for $Osc^1M = TM$ by A. Morimoto [15], partially for the case $k = 2$ by K. Yano and S. Ishihara [17], M. de Léon [4], and completely by R. Miron and the author for the case $k \geq 2$, by using the Sasaki type N- lift (or N-prolongation) G to T^kM of a Riemannian metric g defined on M [7], [10], [13].

In the following, we consider $k = 2$.

The tensor G, (1.7), determines a Riemannian structure on $\widetilde{T^2M} = T^2M \setminus \{0\}$. But G is not homogeneous on the fibres of the second order tangent bundle T^2M, (see [5]). This fact is an inconvenient when an approach to some global problems on the Riemannian space $(\widetilde{T^2M}, G)$ is desired.

In this paper, extending a method of R. Miron from the case $k = 1$ [14], we define by (2.1) a new kind of prolongation $\overset{0}{G}$ to T^2M of the Riemannian metric g. This $\overset{0}{G}$ determines on T^2M a Riemannian structure which is 0-homogeneous on the fibres of T^2M and depends only on g. Some geometrical properties of $\overset{0}{G}$ are studied: the metrical N-linear connection, the Levi-Civita connection etc.

We introduce by (3.1) a natural $f(3,1)-$ structure $\overset{0}{F}$. This depends only on g and is homogeneous. We show that $\overset{0}{F}$ is compatible with $\overset{0}{G}$, that is the pair $(\overset{0}{G}, \overset{0}{F})$ is a Riemannian $f(3,1)-$ structure . We say that it is a geometrical model of the Riemannian space (M, g) with respect to the homogeneous prolongation $\overset{0}{G}$.

1 The Sasaki type N-lift

Let (M, g) be a Riemannian space, M being a real n- dimensional C^∞- manifold and (T^2M, π^2, M) its tangent bundle of second order, [3], [4,5], [7-13]. On a domain $U \subset M$ of a local chart, g has the components $g_{ab}(x)$, $(a, b, c, ... = 1, .., n)$. Then on the domain of chart $(\pi^2)^{-1}(U) \subset T^2M$ we consider the functions $g_{ab}(x, y^{(1)}, y^{(2)}) = g_{ab}(x)$, $\forall (x, y^{(1)}, y^{(2)}) = (u) \in (\pi^2)^{-1}(U)$ and put

$$z^{(2)a} = y^{(2)a} + \frac{1}{2}\gamma^a_{bc}(x)\, y^{(1)b} y^{(1)c}, \tag{1.1}$$

where $\gamma^a_{bc}(x)$ are the Christoffel symbols of the metric g.

Then, $z^{(2)a}$ is globally defined on $\widetilde{T^2M}$ and depends only on the metric g. Moreover, the function

$$L\left(x, y^{(1)}, y^{(2)}\right) = g_{ab}(x)\, z^{(2)a} z^{(2)b} \tag{1.2}$$

is a regular Lagrangian, globally defined on $\widetilde{T^2M}$ and depends only on the metric g. The pair $L^{(2)n} = (M, L)$ is a particular Lagrange space of order 2, having L as fundamental function and $g_{ab}(x)$ as fundamental tensor field. Applying the theory of Lagrange spaces of second order [7-12], we notice that

1^0. The canonical 2- semispray S of the space $L^{(2)n}$ has the coefficients

$$G^a = \frac{1}{3} g^{ab}(x) \left\{ C\left(g_{bc} z^{(2)c}\right) - g_{cd} z^{(2)c} \frac{\partial z^{(2)d}}{\partial y^{(1)b}} \right\}, \tag{1.3}$$

where $C = y^{(1)a}\dfrac{\partial}{\partial x^a} + 2y^{(2)a}\dfrac{\partial}{\partial y^{(1)a}}$.

2^0. The canonical nonlinear connection N has the dual coefficients

$$\begin{aligned} &\underset{1}{M}{}^a{}_b = \gamma^a{}_{bc}(x)\, y^{(1)c} \\ &\underset{2}{M}{}^a{}_b = \frac{1}{2}\left\{ C\left(\gamma^a{}_{bc} y^{(1)c}\right) + \underset{1}{M}{}^a{}_c \underset{1}{M}{}^c{}_b \right\}. \end{aligned} \tag{1.4}$$

3^0. N determines the direct decomposition

$$T_u T^2 M = N_0(u) \oplus N_1(u) \oplus V_2(u),\ \forall u \in T^2 M. \tag{1.5}$$

4^0. The adapted basis to (1.5) is given by $\left(\frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{(1)a}}, \frac{\partial}{\partial y^{(2)a}}\right)$ and its dual basis is $\left(dx^a, \delta y^{(1)a}, \delta y^{(2)a}\right)$, where

$$\left\{ \begin{aligned} \frac{\delta}{\delta x^a} &= \frac{\partial}{\partial x^a} - \underset{1}{N}{}^c{}_a \frac{\partial}{\partial y^{(1)c}} - \underset{2}{N}{}^c{}_a \frac{\partial}{\partial y^{(2)c}} \\ \frac{\delta}{\delta y^{(1)a}} &= \frac{\partial}{\partial y^{(1)a}} - \underset{1}{N}{}^c{}_a \frac{\partial}{\partial y^{(2)c}} \end{aligned} \right. \tag{1.6}$$

and

$$\left\{ \begin{aligned} \delta y^{(1)a} &= dy^{(1)a} + \underset{1}{M}{}^a{}_c dx^c \\ \delta y^{(2)a} &= dy^{(2)a} + \underset{1}{M}{}^a{}_c dy^{(1)c} + \underset{2}{M}{}^a{}_c dx^c \end{aligned} \right. \tag{1.6'}$$

We know that

$$\underset{1}{N}{}^a{}_b = \underset{1}{M}{}^a{}_b,\ \underset{2}{N}{}^a{}_b = \underset{2}{M}{}^a{}_b - \underset{1}{M}{}^a{}_f \underset{1}{M}{}^f{}_b,\ (\underset{2}{M}{}^a{}_b = \underset{2}{N}{}^a{}_b + \underset{1}{N}{}^a{}_f \underset{1}{N}{}^f{}_b). \tag{1.6''}$$

The Sasaki type N- lift of g to T^2M is defined by

$$G(u) = g_{ab}(x)\, dx^a \otimes dx^b + g_{ab}(x)\, \delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab}(x)\, \delta y^{(2)a} \otimes \delta y^{(2)b}, \forall u \in \widetilde{T^2M}. \tag{1.7}$$

The following properties follow:

5^0. G is globally defined on $\widetilde{T^2M}$.

6^0. G is a Riemannian metric on $\widetilde{T^2M}$.

7^0. G is not homogeneous on the fibres of T^2M . Namely, for the homothety

$$h_t : (x, y^{(1)}, y^{(2)}) \to (x, ty^{(1)}, t^2 y^{(2)}) \quad \forall y \in \mathbb{R}^*_+,$$

we get

$$(G \circ h_t)(u) = g_{ab}(x) dx^a \otimes dx^b + t^2 g_{ab}(x) \delta y^{(1)a} \otimes \delta y^{(1)b} + t^4 g_{ab}(x) \delta y^{(2)a} \otimes \delta y^{(2)b} \neq G(u).$$

Let us consider the $\mathcal{F}\left(\widetilde{T^2M}\right)$- linear mapping $F : \mathcal{X}\left(\widetilde{T^2M}\right) \to \mathcal{X}\left(\widetilde{T^2M}\right)$ given in the adapted basis by

$$F\left(\frac{\delta}{\delta x^a}\right) = -\frac{\delta}{\delta y^{(1)a}}, F\left(\frac{\delta}{\delta y^{(1)a}}\right) = \frac{\delta}{\delta x^a}, F\left(\frac{\partial}{\partial y^{(2)a}}\right) = 0,\ \ a = \overline{1,n}. \tag{1.8}$$

It follows that:

8^0. F is globally defined on $\widetilde{T^2M}$ and it is a d- tensor field of type $(1,1)$.

9^0. F is an $f(3,1)-$ structure: $F^3 + F = 0$.

10^0. F depends only on g.

11^0. The pair (G, F) is a Riemannian $f(3,1)-$ structure on $\widetilde{T^2M}$: $G(FX, Y) =$

$-G\left(X,FY\right), \forall X,Y\in\mathcal{X}\left(\widetilde{T^2M}\right)$.

Consequently, we get

Theorem 1.1. *The space* $\left(T^2M,G,F\right)$ *is a Riemannian* $f(3,1)-$ *space depending only on the Riemannian metric* g.

The previous space, called the geometrical model on T^2M of the Riemannian space (M,g) is important in the study of the geometry of the given Riemannian space (M,g).

2 The homogeneous prolongation of the Riemannian metric g

We can eliminate the inconvenient of the Sasaki type N-lift G given by the property 7^0 introducing a new kind of prolongation to T^2M of the Riemannian metric g.

Definition 2.1. *We call homogeneous prolongation of the Riemannian metric* g *the following tensor field on* $\widetilde{T^2M}$:

$$\begin{gathered}\overset{0}{G}\left(u\right)=g_{ab}\left(x\right)dx^a\otimes dx^b+\frac{k^2}{\left\|y^{(1)}\right\|^2}g_{ab}\left(x\right)\delta y^{(1)a}\otimes\delta y^{(1)b}+\\+\frac{k^4}{\left\|y^{(1)}\right\|^4}g_{ab}\left(x\right)\delta y^{(2)a}\otimes\delta y^{(2)b},\forall u\in\widetilde{T^2M}\end{gathered}\tag{2.1}$$

where $k>0$ *is a constant and* $\left\|y^{(1)}\right\|^2$ *is the square of the norm of the first Liouville vector field:*

$$\left\|y^{(1)}\right\|^2=g_{ab}\left(x\right)y^{(1)a}y^{(1)b}.\tag{2.2}$$

We get, without difficulties:

Theorem 2.1. *The following properties hold:*

1^0. *The pair* $(\widetilde{T^2M},\overset{0}{G})$ *is a Riemannian space depending only on the metric* g.

2^0. $\overset{0}{G}$ *is 0-homogeneous on the fibres of the second tangent bundle* T^2M.

3^0. *The distributions* N_0,N_1 *and* V_2 *are orthogonal, in pairs, with respect to* $\overset{0}{G}$.

We can write $\overset{0}{G}$ in the form

$$\overset{0}{G}=\overset{0}{G}{}^H+\overset{0}{G}{}^{V_1}+\overset{0}{G}{}^{V_2},\tag{2.3}$$

where

$$\overset{0}{G}{}^H=g_{ab}\left(x\right)dx^a\otimes dx^b,\overset{0}{G}{}^{V_1}=\underset{(1)}{g}{}_{ab}\delta y^{(1)a}\otimes\delta y^{(1)b},\ \overset{0}{G}{}^{V_2}=\underset{(2)}{g}{}_{ab}\delta y^{(2)a}\otimes\delta y^{(2)b}\tag{2.4}$$

and

$$\underset{(1)}{g}{}_{ab}=\frac{k^2}{\left\|y^{(1)}\right\|^2}g_{ab},\ \underset{(2)}{g}{}_{ab}=\frac{k^4}{\left\|y^{(1)}\right\|^4}g_{ab}.\tag{2.5}$$

As usually, let us denote

$$\partial_a = \frac{\partial}{\partial x^a}, \dot{\partial}_{1a} = \frac{\partial}{\partial y^{(1)a}},\ \dot{\partial}_{2a} = \frac{\partial}{\partial y^{(2)a}}$$

and from now on we denote the adapted basis to (1.5) by $(\delta_a, \delta_{1a}, \dot{\partial}_{2a})$.

In order to study the geometry of the Riemannian space $(\widetilde{T^2M}, \overset{0}{G})$, we can apply the theory of the (h, v_1, v_2)-Riemannian metrics on T^2M given recently, [3], by the author.

A linear connection D on T^2M is called a metrical N-linear connection with respect to $\overset{0}{G}$ if $D\overset{0}{G} = 0$ and D preserves by parallelism the vertical distributions N_1 and V_2.

We can easily prove the existence of the metrical N-linear connections in the adapted basis. To this aim we represent a linear connection D in the adapted basis in the following form

$$\begin{aligned}
&D_{\delta_c}\delta_b = \underset{(00)}{\overset{0}{L}}{}^{a}{}_{bc}\delta_a + \underset{(00)}{\overset{1}{L}}{}^{a}{}_{bc}\delta_{1a} + \underset{(00)}{\overset{2}{L}}{}^{a}{}_{bc}\dot{\partial}_{2a}, \quad D_{\delta_c}\delta_{1b} = \underset{(10)}{\overset{0}{L}}{}^{a}{}_{bc}\delta_a + \underset{(10)}{\overset{1}{L}}{}^{a}{}_{bc}\delta_{1a} + \underset{(10)}{\overset{2}{L}}{}^{a}{}_{bc}\dot{\partial}_{2a},\\
&D_{\delta_c}\dot{\partial}_{2b} = \underset{(20)}{\overset{0}{L}}{}^{a}{}_{bc}\delta_a + \underset{(20)}{\overset{1}{L}}{}^{a}{}_{bc}\delta_{1a} + \underset{(20)}{\overset{2}{L}}{}^{a}{}_{bc}\dot{\partial}_{2a}, \quad D_{\delta_{1c}}\delta_b = \underset{(01)}{\overset{0}{C}}{}^{a}{}_{bc}\delta_a + \underset{(01)}{\overset{1}{C}}{}^{a}{}_{bc}\delta_{1a} + \underset{(01)}{\overset{2}{C}}{}^{a}{}_{bc}\dot{\partial}_{2a},\\
&D_{\delta_{1c}}\delta_{1b} = \underset{(11)}{\overset{0}{C}}{}^{a}{}_{bc}\delta_a + \underset{(11)}{\overset{1}{C}}{}^{a}{}_{bc}\delta_{1a} + \underset{(11)}{\overset{2}{C}}{}^{a}{}_{bc}\dot{\partial}_{2a}, \quad D_{\delta_{1c}}\dot{\partial}_{2b} = \underset{(21)}{\overset{0}{C}}{}^{a}{}_{bc}\delta_a + \underset{(21)}{\overset{1}{C}}{}^{a}{}_{bc}\delta_{1a} + \underset{(21)}{\overset{2}{C}}{}^{a}{}_{bc}\dot{\partial}_{2a},
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&D_{\dot{\partial}_{2c}}\delta_b = \underset{(02)}{\overset{0}{C}}{}^{a}{}_{bc}\delta_a + \underset{(02)}{\overset{1}{C}}{}^{a}{}_{bc}\delta_{1a} + \underset{(02)}{\overset{2}{C}}{}^{a}{}_{bc}\dot{\partial}_{2a}, \quad D_{\dot{\partial}_{2c}}\delta_{1b} = \underset{(12)}{\overset{0}{C}}{}^{a}{}_{bc}\delta_a + \underset{(12)}{\overset{1}{C}}{}^{a}{}_{bc}\delta_{1a} + \underset{(12)}{\overset{2}{C}}{}^{a}{}_{bc}\dot{\partial}_{2a},\\
&D_{\dot{\partial}_{2c}}\dot{\partial}_{2b} = \underset{(22)}{\overset{0}{C}}{}^{a}{}_{bc}\delta_a + \underset{(22)}{\overset{1}{C}}{}^{a}{}_{bc}\delta_{1a} + \underset{(22)}{\overset{2}{C}}{}^{a}{}_{bc}\dot{\partial}_{2a}.
\end{aligned}$$

The systems of functions

$$\left(\underset{(00)}{\overset{\alpha}{L}}{}^{a}{}_{bc}, \underset{(10)}{\overset{\alpha}{L}}{}^{a}{}_{bc}, \underset{(20)}{\overset{\alpha}{L}}{}^{a}{}_{bc}, \underset{(01)}{\overset{\alpha}{C}}{}^{a}{}_{bc}, \underset{(11)}{\overset{\alpha}{C}}{}^{a}{}_{bc}, \underset{(21)}{\overset{\alpha}{C}}{}^{a}{}_{bc}, \underset{(02)}{\overset{\alpha}{C}}{}^{a}{}_{bc}, \underset{(12)}{\overset{\alpha}{C}}{}^{a}{}_{bc}, \underset{(22)}{\overset{\alpha}{C}}{}^{a}{}_{bc}\right),$$

$(\alpha = 0, 1, 2)$, are the coefficients of D and

$$\left(\underset{(00)}{\overset{0}{L}}{}^{a}{}_{bc}, \underset{(10)}{\overset{1}{L}}{}^{a}{}_{bc}, \underset{(20)}{\overset{2}{L}}{}^{a}{}_{bc}, \underset{(01)}{\overset{0}{C}}{}^{a}{}_{bc}, \underset{(11)}{\overset{1}{C}}{}^{a}{}_{bc}, \underset{(21)}{\overset{2}{C}}{}^{a}{}_{bc}, \underset{(02)}{\overset{0}{C}}{}^{a}{}_{bc}, \underset{(12)}{\overset{1}{C}}{}^{a}{}_{bc}, \underset{(22)}{\overset{2}{C}}{}^{a}{}_{bc}\right)$$

are the coefficients of an N-linear connection $D\Gamma(N)$. Also, the coefficients of $D\Gamma(N)$ will be denoted with $\left(\underset{(00)}{\overset{H}{L}}{}^{a}{}_{bc}, \underset{(\beta 0)}{\overset{V_\beta}{L}}{}^{a}{}_{bc}, \underset{(01)}{\overset{H}{C}}{}^{a}{}_{bc}, \underset{(\beta 1)}{\overset{V_\beta}{C}}{}^{a}{}_{bc}, \underset{(02)}{\overset{H}{C}}{}^{a}{}_{bc}, \underset{(\beta 2)}{\overset{V_\beta}{C}}{}^{a}{}_{bc}\right)$, $(\beta = 1, 2)$.

It is not difficult to prove:

Theorem 2.2. *There exist metrical N-linear connections $D\Gamma(N)$ on $\widetilde{T^2M}$, with respect to the homogeneous prolongation $\overset{0}{G}$, which depend only on the metric tensor g. One of them has the following coefficients:*

$$\underset{(00)}{\overset{\beta}{L}}{}^{a}{}_{bc} = \underset{(10)}{\overset{\delta}{L}}{}^{a}{}_{bc} = \underset{(20)}{\overset{\varepsilon}{L}}{}^{a}{}_{bc} = \underset{(0\beta)}{\overset{\alpha}{C}}{}^{a}{}_{bc} = \underset{(11)}{\overset{\delta}{C}}{}^{a}{}_{bc} = \underset{(21)}{\overset{\varepsilon}{C}}{}^{a}{}_{bc} = \underset{(12)}{\overset{\alpha}{C}}{}^{a}{}_{bc} = \underset{(22)}{\overset{\alpha}{C}}{}^{a}{}_{bc} = 0,$$

$$(\alpha = 0, 1, 2;\ \beta = 1, 2;\ \delta = 0, 2;\ \varepsilon = 0, 1)$$

$$\begin{aligned}
&\underset{(00)}{\overset{H}{L}}{}^{a}{}_{bc} = \underset{(10)}{\overset{V_1}{L}}{}^{a}{}_{bc} = \underset{(20)}{\overset{V_2}{L}}{}^{a}{}_{bc} = \gamma^{a}_{bc}\left(x\right)\\
&\underset{(11)}{\overset{V_1}{C}}{}^{a}{}_{bc} = -\frac{1}{\left\|y^{(1)}\right\|^{2}}\left(\delta^{a}_{b}y^{(1)}_{c} + \delta^{a}_{c}y^{(1)}_{b} - g_{bc}y^{(1)a}\right)\\
&\underset{(21)}{\overset{V_2}{C}}{}^{a}{}_{bc} = -\frac{2}{\left\|y^{(1)}\right\|^{2}}\left(\delta^{a}_{b}y^{(1)}_{c} + \delta^{a}_{c}y^{(1)}_{b} - g_{bc}y^{(1)a}\right)
\end{aligned} \tag{2.7}$$

where $y^{(1)}_{a} = g_{ab}\left(x\right)y^{(1)b}$.

The simplicity of this metrical N-linear connection and the fact that it is determined only by the Riemannian metric g, are reasons to call it the **canonical metrical** N-linear connection of the space $(\widetilde{T^{2}M}, \overset{0}{G})$.

The structure equations of the canonical metrical N-linear connection are very simple, too, [2], [3]. Its structure forms are as follows:

$$\underset{(0)}{\omega}{}^{a}{}_{b} = \gamma^{a}_{bc}\left(x\right)dx^{c},\ \underset{(\beta)}{\omega}{}^{a}{}_{b} = \gamma^{a}_{bc}\left(x\right)dx^{c} + \underset{(\beta 1)}{\overset{\beta}{C}}{}^{a}{}_{bc}\delta y^{(1)c},\ (\beta = 1,2). \tag{2.8}$$

Theorem 2.3. *The structure equations of the canonical metrical* N*-linear connection* D *of the Riemann space* $(\widetilde{T^{2}M}, \overset{0}{G})$, $\overset{0}{G}$ *being the homogeneous prolongation of the metric* g, *are given by:*

$$\begin{cases}
d\left(dx^{a}\right) - dx^{b}\wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} = -\underset{(\alpha)}{\overset{0}{\Omega}}{}^{a}\\
d\left(\delta y^{(\beta)a}\right) - \delta y^{(\beta)b}\wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} = -\underset{(\alpha)}{\overset{\beta}{\Omega}}{}^{a}, (\alpha = 0,1,2; \beta = 1,2)
\end{cases} \tag{2.9}$$

$$d\,\underset{(\alpha)}{\omega}{}^{a}{}_{b} - \underset{(\alpha)}{\omega}{}^{c}{}_{b}\wedge \underset{(\alpha)}{\omega}{}^{a}{}_{c} = -\underset{(\alpha)}{\Omega}{}^{a}{}_{b}, (\alpha = 0,1,2) \tag{2.10}$$

where the 2-forms of torsion $\underset{(\alpha)}{\overset{0}{\Omega}}{}^{a}$ *and* $\underset{(\alpha)}{\overset{\beta}{\Omega}}{}^{a}\ (\alpha = 0,1,2; \beta = 1,2)$ *are*

$$\begin{cases}
\underset{(0)}{\overset{0}{\Omega}}{}^{a} = 0,\\
\underset{(1)}{\overset{0}{\Omega}}{}^{a} = \underset{(11)}{\overset{1}{C}}{}^{a}{}_{bc}dx^{b}\wedge \delta y^{(1)c},\ \underset{(2)}{\overset{0}{\Omega}}{}^{a} = \underset{(21)}{\overset{2}{C}}{}^{a}{}_{bc}dx^{b}\wedge \delta y^{(1)c},
\end{cases} \tag{2.11$_1$}$$

$$\underset{(0)}{\overset{1}{\Omega}}{}^{a} = \underset{(1)}{\overset{1}{\Omega}}{}^{a} = \underset{(2)}{\overset{1}{\Omega}}{}^{a} = \frac{1}{2}\underset{(01)}{R}{}^{a}{}_{bc}dx^{b}\wedge dx^{c}, \tag{2.11$_2$}$$

$$\begin{cases}
\underset{(0)}{\overset{2}{\Omega}}{}^{a} = \frac{1}{2}\underset{(02)}{R}{}^{a}{}_{bc}dx^{b}\wedge dx^{c} + \underset{(12)}{B}{}^{a}{}_{bc}dx^{b}\wedge \delta y^{(1)c},\\
\underset{(1)}{\overset{2}{\Omega}}{}^{a} = \frac{1}{2}\underset{(02)}{R}{}^{a}{}_{bc}dx^{b}\wedge dx^{c} + \underset{(12)}{B}{}^{a}{}_{bc}dx^{b}\wedge \delta y^{(1)c} - \underset{(11)}{\overset{1}{C}}{}^{a}{}_{bc}\delta y^{(1)b}\wedge \delta y^{(2)c},\\
\underset{(2)}{\overset{2}{\Omega}}{}^{a} = \frac{1}{2}\underset{(02)}{R}{}^{a}{}_{bc}dx^{b}\wedge dx^{c} + \underset{(12)}{B}{}^{a}{}_{bc}dx^{b}\wedge \delta y^{(1)c} - \underset{(21)}{\overset{2}{C}}{}^{a}{}_{bc}\delta y^{(1)b}\wedge \delta y^{(2)c},
\end{cases} \tag{2.11$_3$}$$

and the 2-forms of curvature $\underset{(0)}{\Omega}{}^{a}{}_{b}$, $(\alpha = 0, 1, 2)$ *are*

$$\begin{cases} \underset{(0)}{\Omega}{}^{a}{}_{b} = \dfrac{1}{2}\underset{(00)}{R}{}_{b}{}^{a}{}_{cd}dx^{c}\wedge dx^{b} \\ \underset{(1)}{\Omega}{}^{a}{}_{b} = \dfrac{1}{2}\underset{(01)}{R}{}_{b}{}^{a}{}_{cd}dx^{c}\wedge dx^{d} + \underset{(11)}{P}{}_{b}{}^{a}{}_{cd}dx^{c}\wedge\delta y^{(1)d} + \dfrac{1}{2}\underset{(11)}{S}{}_{b}{}^{a}{}_{cd}\delta y^{(1)c}\wedge\delta y^{(1)d} \\ \underset{(2)}{\Omega}{}^{a}{}_{b} = \dfrac{1}{2}\underset{(02)}{R}{}_{b}{}^{a}{}_{cd}dx^{c}\wedge dx^{d} + \underset{(12)}{P}{}_{b}{}^{a}{}_{cd}dx^{c}\wedge\delta y^{(1)d} + \dfrac{1}{2}\underset{(12)}{S}{}_{b}{}^{a}{}_{cd}\delta y^{(1)c}\wedge\delta y^{(1)d} \end{cases} \tag{2.12}$$

where $\underset{(00)}{R}{}_{b}{}^{a}{}_{cd}$ *is the curvature tensor of the metric* g, $\underset{(12)}{P}{}_{b}{}^{a}{}_{cd}$ *are the* $(v\,h)\,v_{\beta}$ *curvature tensors of* D, $\underset{(12)}{S}{}_{b}{}^{a}{}_{cd}$ *are* $(v_{\beta}\,v_{\beta})\,v_{1}$ *curvature tensors of* D, *i.e.*

$$\begin{aligned} &\underset{(00)}{R}{}_{b}{}^{a}{}_{cd} = \partial_{d}\gamma^{a}_{bc} - \partial_{c}\gamma^{a}_{bd} + \gamma^{f}_{bc}\gamma^{a}_{fd} - \gamma^{f}_{bd}\gamma^{a}_{fc}, \\ &\underset{(1\beta)}{P}{}_{b}{}^{a}{}_{cd} = -\left(\partial_{1c}\overset{\beta}{\underset{(\beta1)}{C}}{}^{a}{}_{bd} - \overset{\beta}{\underset{(\beta0)}{L}}{}^{f}{}_{bc}\overset{\beta}{\underset{(\beta1)}{C}}{}^{a}{}_{fd} - \overset{\beta}{\underset{(\beta0)}{L}}{}^{f}{}_{dc}\overset{\beta}{\underset{(\beta1)}{C}}{}^{a}{}_{bf} + \overset{\beta}{\underset{(\beta0)}{L}}{}^{a}{}_{fc}\overset{\beta}{\underset{(\beta1)}{C}}{}^{f}{}_{bd}\right), (\beta = 1,2), \\ &\underset{(1\beta)}{S}{}_{b}{}^{a}{}_{cd} = \partial_{1d}\overset{\beta}{\underset{(\beta1)}{C}}{}^{a}{}_{bc} - \partial_{1c}\overset{\beta}{\underset{(\beta1)}{C}}{}^{a}{}_{bd} + \overset{\beta}{\underset{(\beta1)}{C}}{}^{f}{}_{bc}\overset{\beta}{\underset{(\beta1)}{C}}{}^{a}{}_{fd} - \overset{\beta}{\underset{(\beta1)}{C}}{}^{f}{}_{bd}\overset{\beta}{\underset{(\beta1)}{C}}{}^{a}{}_{fc}, (\beta = 1,2), \\ &\underset{(01)}{R}{}_{b}{}^{a}{}_{cd} = \underset{(00)}{R}{}_{b}{}^{a}{}_{cd} + \overset{1}{\underset{(11)}{C}}{}^{a}{}_{bf}\underset{(01)}{R}{}^{f}{}_{cd}, \\ &\underset{(02)}{R}{}_{b}{}^{a}{}_{cd} = \underset{(00)}{R}{}_{b}{}^{a}{}_{cd} + \overset{2}{\underset{(21)}{C}}{}^{a}{}_{bf}\underset{(02)}{R}{}^{f}{}_{cd}, \\ &\underset{(01)}{R}{}^{a}{}_{bc}\left(x, y^{(1)}, y^{(2)}\right) = y^{(1)d}\underset{(00)}{R}{}_{d}{}^{a}{}_{bc} \end{aligned} \tag{2.13}$$

and $\underset{(01)}{R}{}^{f}{}_{cd}$, $\underset{(12)}{B}{}^{a}{}_{bc}$ *are the following d- tensor fields on* $T^{2}M$:

$$\begin{aligned} \underset{(02)}{R}{}^{a}{}_{bc} &= \delta_{c}\underset{2}{N}{}^{a}{}_{b} - \delta_{b}\underset{2}{N}{}^{a}{}_{c} + \underset{1}{N}{}^{a}{}_{f}\underset{(01)}{R}{}^{f}{}_{bc}, \\ \underset{(12)}{B}{}^{a}{}_{bc} &= \delta_{1c}\underset{2}{N}{}^{a}{}_{b} - \delta_{b}\underset{1}{N}{}^{a}{}_{c} + \underset{1}{N}{}^{a}{}_{f}\delta_{1c}\underset{1}{N}{}^{f}{}_{b}. \end{aligned} \tag{2.14}$$

Finally, by a straightforward calculus we can determine the Levi-Civita connection of the Riemannian metric $\overset{0}{G}$.

Theorem 2.4. *In the adapted basis the coefficients* $\left(\overset{0}{\underset{(00)}{L}}{}^{a}{}_{bc}, \ldots, \overset{2}{\underset{(22)}{C}}{}^{a}{}_{bc}\right)$ *from* (2.6) *of the Levi-Civita connection of the Riemannian metric* $\overset{0}{G}$ *from* (2.1) *are as follows:*

$$\begin{aligned} &\overset{\beta}{\underset{(0\beta)}{C}}{}^{a}{}_{bc} = \overset{\delta}{\underset{(\beta\beta)}{C}}{}^{a}{}_{bc} = \overset{\varepsilon}{\underset{(21)}{C}}{}^{a}{}_{bc} = \overset{1}{\underset{(12)}{C}}{}^{a}{}_{bc} = 0, \quad (\beta = 1,2;\ \delta = 0,2) \\ &\overset{0}{\underset{(00)}{L}}{}^{a}{}_{bc} = \overset{1}{\underset{(10)}{L}}{}^{a}{}_{bc} = \overset{2}{\underset{(20)}{L}}{}^{a}{}_{bc} = \gamma^{a}{}_{bc}(x), \\ &\overset{\beta}{\underset{(00)}{L}}{}^{a}{}_{bc} = -\frac{1}{2}\underset{(0\beta)}{R}{}^{a}{}_{bc}, \ \overset{0}{\underset{(\beta0)}{L}}{}^{a}{}_{bc} = \overset{0}{\underset{(0\beta)}{C}}{}^{a}{}_{cb} = \frac{1}{2}g^{ad}\underset{(\beta)}{g}{}_{bf}\underset{(0\beta)}{R}{}^{f}{}_{dc}, \quad (\beta = 1,2), \end{aligned}$$

$$\underset{(10)}{\overset{2}{L}}{}^{a}{}_{bc} = -\underset{(01)}{\overset{2}{C}}{}^{a}{}_{cb} = \frac{1}{2}\underset{(12)}{B}{}^{a}{}_{cb},\ \underset{(12)}{\overset{2}{C}}{}^{a}{}_{bc} = \frac{1}{2}g^{ad}\underset{(2)}{g}{}_{fc}\underset{(12)}{B}{}^{f}{}_{db},$$

$$\underset{(20)}{\overset{1}{L}}{}^{a}{}_{bc} = -\underset{(02)}{\overset{1}{C}}{}^{a}{}_{cb} = -\frac{1}{2}\underset{(1)}{g}{}^{ad}\underset{(2)}{g}{}_{bf}\underset{(12)}{B}{}^{f}{}_{cd},$$

$$\underset{(11)}{\overset{1}{C}}{}^{a}{}_{bc} = -\frac{1}{\left\|y^{(1)}\right\|^{2}}\left(\delta^{a}_{b}y^{(1)}_{c} + \delta^{a}_{c}y^{(1)}_{b} - g_{bc}y^{(1)a}\right),$$

$$\underset{(21)}{\overset{2}{C}}{}^{a}{}_{bc} = \underset{(12)}{\overset{2}{C}}{}^{a}{}_{cb} = -2\underset{(2)}{\overset{2}{O}}{}^{ad}_{fb}\frac{1}{\left\|y^{(1)}\right\|^{2}}\left(\delta^{f}_{d}y^{(1)}_{c} + \delta^{f}_{c}y^{(1)}_{d} - g_{dc}y^{(1)a}\right)$$

$$\underset{(22)}{\overset{1}{C}}{}^{a}{}_{bc} = -2\underset{(1)}{g}{}^{ad}\underset{(2)}{g}{}_{ab}\underset{(2)}{\overset{2}{O}}{}^{fe}_{hc}\frac{1}{\left\|y^{(1)}\right\|^{2}}\left(\delta^{h}_{e}y^{(1)}_{d} + \delta^{h}_{d}y^{(1)}_{e} - g_{de}y^{(1)h}\right)$$

where $\underset{(2)}{\overset{2}{O}}{}^{ad}_{fb} = \frac{1}{2}\left(\delta^{a}_{f}\delta^{d}_{b} + \underset{(2)}{g}{}_{ab}\underset{(2)}{g}{}^{ad}\right).$

Of course, the structure equations of the Levi-Civita connection can be written without difficulties.

3 The Riemannian $f(3,1)-$ structure $(\overset{0}{G}, \overset{0}{F})$

The $f(3,1)-$ structure F defined in (1.8) has not the property of homogeneity. The $\mathcal{F}\left(\widetilde{T^2M}\right)$- linear mapping $F : \mathcal{X}\left(\widetilde{T^2M}\right) \to \mathcal{X}\left(\widetilde{T^2M}\right)$, applies the 1-homogeneous vector fields δ_a into the 0-homogeneous vector fields δ_{1a}, $(a = 1, ...n)$.

To remedy this, we consider the $\mathcal{F}\left(\widetilde{T^2M}\right)$- linear mapping $\overset{0}{F} : \mathcal{X}\left(\widetilde{T^2M}\right) \to \mathcal{X}\left(\widetilde{T^2M}\right)$, given in the adapted basis by

$$\overset{0}{F}(\delta_a) = -\frac{\left\|y^{(1)}\right\|}{k}\delta_{1a},\ \overset{0}{F}(\delta_{1a}) = \frac{k}{\left\|y^{(1)}\right\|}\delta_a,\ \overset{0}{F}(\partial_{2a}) = 0, (a = 1, ..., n). \tag{3.1}$$

By direct calculus, we can prove:

Theorem 3.1. *$\overset{0}{F}$ has the following properties:*

1^0. *$\overset{0}{F}$ is a tensor field of type $(1,1)$ on $\widetilde{T^2M}$.*

2^0. *$\overset{0}{F}$ is an $f(3,1)-$ structure on $\widetilde{T^2M}$:*

$$\overset{0}{F}{}^{3} + \overset{0}{F} = 0. \tag{3.2}$$

3^0. *$\overset{0}{F}$ depends only on the metric g.*

4^0. *$\overset{0}{F}$ is homogeneous on the fibres of T^2M.*

5^0. The pair $(\overset{0}{G}, \overset{0}{F})$ is a Riemannian $f(3,1)-$ structure on $\widetilde{T^2M}$: $\overset{0}{G}\left(\overset{0}{F}X, Y\right) = -\overset{0}{G}\left(X, \overset{0}{F}Y\right)$, $\forall X, Y \in \mathcal{X}\left(\widetilde{T^2M}\right)$.

Theorem 3.2. *The Nijenhuis tensor* $\mathcal{N}_{\overset{0}{F}}$ *of* $\overset{0}{F}$ *vanishes on* T^2M *if and only if the Riemannian space* (M,g) *is of constant curvature* $\left(=\frac{1}{k^2}\right)$ *and*

$$\underset{(12)}{B}{}^{a}{}_{bc}=0,\ \underset{(02)}{R}{}^{a}{}_{bc}=0. \tag{3.3}$$

Proof. The Nijenhuis tensor of the $f(3,1)-$ structure $\overset{0}{F}$ is

$$\mathcal{N}_{\overset{0}{F}}(X,Y)=\overset{0}{F}{}^2[X,Y]+[\overset{0}{F}X,\overset{0}{F}Y]-\overset{0}{F}[\overset{0}{F}X,Y]-\overset{0}{F}[X,\overset{0}{F}Y],$$

Using the adapted basis, it comes out that the Nijenhuis tensor field $\mathcal{N}_{\overset{0}{F}}$ of $\overset{0}{F}$ vanishes if and only if we have:

$$\underset{(01)}{R}{}^{a}{}_{bc}=\frac{1}{k^2}\left(y^{(1)}_{b}\delta^a_c-y^{(1)}_{c}\delta^a_b\right),\ \underset{(02)}{R}{}^{a}{}_{bc}=0,\ \underset{(12)}{B}{}^{a}{}_{bc}=0.$$

Taking into account (2.13) and the fact that $y^{(1)}_{a}=g_{ab}(x)\,y^{(1)b}$ the first equality is equivalent to

$$\underset{(00)}{R}{}^{a}{}_{bcd}=\frac{1}{k^2}\left(g_{bc}\delta^a_d-g_{bd}\delta^a_c\right),$$

q.e.d.

Theorem 3.3. *The conditions of normality of* $\overset{0}{F}$ *are as follows:*

$$\begin{gathered}\underset{(01)}{R}{}^{a}{}_{bc}=\frac{1}{a^2}\left(y^{(1)}_{b}\delta^a_c-y^{(1)}_{c}\delta^a_b\right),\quad \underset{(02)}{R}{}^{a}{}_{bc}=0,\\ \underset{(12)}{B}{}^{a}{}_{bc}+\sum_{a=1}^{n}\underset{(12)}{B}{}^{a}{}_{bc}=0,\quad \sum_{a=1}^{n}\gamma^a_{bc}(x)=0.\end{gathered} \tag{3.4}$$

Proof. The condition of normality of $\overset{0}{F}$ is

$$\mathcal{N}_{\overset{0}{F}}(X,Y)+\sum_{a=1}^{n}d\left(\delta y^{(2)a}\right)(X,Y)=0,\ \forall X,Y\in\mathcal{X}\left(\widetilde{T^2M}\right).$$

Taking into account that we have

$$d\left(\delta y^{(2)a}\right)(X,Y)=\left\{\frac{1}{2}\underset{(02)}{R}{}^{a}{}_{bc}dx^c+\underset{(12)}{B}{}^{a}{}_{bc}\delta y^{(1)c}+\gamma^a_{bc}(x)\,\delta y^{(2)c}\right\}\wedge dx^b,$$

in the adapted basis, we obtain the equalities (3.4). q.e.d.

The space $(\widetilde{T^2M},\overset{0}{G})$ is called the homogeneous geometrical model of the Riemannian space (M,g) with respect to the homogeneous prolongation $\overset{0}{G}$. It could be used in a gauge theory of second order on the lines from [1], [16].

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