LIBERTAS MATHEMATICA, vol XXIV (2004)

On a Framed $f-$Structure on Tangent Manifold of a Riemannian Space

Adrian SANDOVICI and Victor BLĂNUŢĂ

Abstract. We show that on the tangent bundle of a Riemannian space a natural framed $f-$structure of corank 2 can be introduced. Also, we obtain two results regarding the existence of $d-$connections compatible with this framed $f-$structure.

Mathematics Subject Classifications 2000: 53C60.

Keywords: Riemannian space, tangent manifold, framed $f-$structure, metrical almost 2-π structure, $d-$connection compatible with a framed $f-$structure.

1 Introduction

Let $R^{(n)} = (M, \gamma)$ be a Riemannian space with a smooth, real manifold M and a Riemannian structure γ. Here $x = (x^i)$ are coordinates on M and $(x, y) = (x^i, y^i)$ are coordinates on the tangent manifold TM projected on M by τ. The indices $i, j, k...$ will run from 1 to $n = \dim M$ and the Einstein convention on summation is implied. The geometrical objects on TM whose local components change like on M will be called $d-$objects as in [10]. The kernel of the differential $\tau^T : TTM \to TM$ is a vector subbundle of TTM called the vertical distribution on TM. The local vector fields $\left\{\dfrac{\partial}{\partial y^i}\right\}$ determine a local frame in VTM. A nonlinear connection in the tangent bundle $\tau : TM \to M$ is a distribution HTM on TM supplementary to the vertical distribution, that is:

$$TTM = HTM \oplus VTM \tag{1.1}$$

The position of the subspace H_uTM, $u \in TM$ can be given by n local vector fields $\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - N_i^k(x,y)\dfrac{\partial}{\partial y^k}$. The real differentiable functions $\left(N_i^k(x,y)\right)$ completely determine a nonlinear connection which will be denoted simply by N. For example we can take $N_j^i(x,y) := \gamma_{jk}^i(x)y^k$, where $\gamma_{jk}^i(x)$ are the Christoffel symbols of the Levi–Civita connection. Therefore a nonlinear connection N determines a basis $\left(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right)$ adapted to the decomposition (1.1).

We recall that the Sasaki lift of the Riemannian structure γ on TM is as follows:

$$G_S = \gamma_{ij}(x)dx^i \otimes dx^j + \gamma_{ij}(x)\delta y^i \otimes \delta y^j, \tag{1.2}$$

where $\delta y^i = dy^i + N^i_k(x,y)\,dx^k$.

Next we consider a $(h,v)-$ metrical structure on TM given by:

$$G(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + h_{ij}(x,y)\delta y^i \otimes \delta y^j \tag{1.3}$$

for:

$$\begin{aligned} g_{ij}(x,y) &= \frac{a^2}{F^2}\gamma_{ij}(x) + \frac{b^2-a^2}{F^4}y_i y_j \\ h_{ij}(x,y) &= \frac{c^2}{F^2}\gamma_{ij}(x) + \frac{d^2-c^2}{F^4}y_i y_j \end{aligned} \tag{1.4}$$

where $F^2 = \gamma_{ij}(x)y^i y^j, y_i = \gamma_{ij}(x)y^j$ and $a, b, c, d : \mathrm{Im}(F^2) \subseteq \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are differentiable functions with $b \geq a > 0$, $d \geq c > 0$.

The study of this $(h,v)-$ metrical structure was materialized in [11] and [12].

Now, let us assume TM is endowed with a nonlinear connection given by the local coefficients $N^i_j(x,y) = \gamma^i_{jk}(x)y^k$.

Definition 1.1. *A d-connection D on TM is called compatible with the metrical structure G if it satisfies the condition:*

$$D_X G = 0, \quad \forall X \in \chi(TM) \tag{1.5}$$

In the adapted basis, any $d-$connection on TM can be represented in the following way:

$$\begin{aligned} D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j} &= \overset{(H)}{F}{}^i_{jk}\frac{\delta}{\delta x^i}, \quad D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial y^j} = \overset{(V)}{F}{}^i_{jk}\frac{\partial}{\partial y^i} \\ D_{\frac{\partial}{\partial y^k}}\frac{\delta}{\delta x^j} &= \overset{(H)}{C}{}^i_{jk}\frac{\delta}{\delta x^i}, \quad D_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j} = \overset{(V)}{C}{}^i_{jk}\frac{\partial}{\partial y^i} \end{aligned} \tag{1.6}$$

in which the system of functions $(\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$ represents the local coefficients of the above d-connection D.

Recall that an almost 2–π structure is a tensor field of type (1,1) on TM, which, in adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ is given by:

$$\Phi_S\left(\frac{\delta}{\delta x^i}\right) = -\lambda \cdot \frac{\partial}{\partial y^i}, \quad \Phi_S\left(\frac{\partial}{\partial y^i}\right) = \lambda \cdot \frac{\delta}{\delta x^i}, \quad \lambda \in C \tag{1.7}$$

Also, a metrical almost 2–π structure is a pair (G_S, Φ_S) on TM for which:

$$\frac{1}{\lambda^2}G_S(\Phi_S X, \Phi_S Y) = G_S(X,Y), \quad \forall X, Y \in \chi(TM) \tag{1.8}$$

and the 2-form $\Omega_S(X,Y) = G_S(\,\Phi_S(X), Y)$ is closed.

The pair $(G,\ \Phi_S)$, where the metrical structure G is defined by (1.3), (1.4), is not a metrical almost 2–π structure. In [11] we proved that there is a class of almost 2-π structures Φ such that the pair $(G,\ \Phi)$ is a metrical almost 2–π structure. In fact we proved that the pairs $(G_{a,b,c,d},\ \Phi_{a,b,c,d})$ and $(G_{a,a,c,c},\ \Phi_{a,a,c,c})$ are metrical almost 2–π structures on the tangent bundle. The almost 2–π structures $\Phi_{a,b,c,d}$ and $\Phi_{a,a,c,c}$ are given by:

$$\Phi_{a,b,c,d} = \lambda A^k_i \frac{\partial}{\partial y^k} \otimes dx^i + \lambda B^k_i \frac{\delta}{\delta x^k} \otimes \delta y^i \tag{1.9}$$

and:

$$\Phi_{a,a,c,c} = \lambda \overset{\smile}{A}{}^k_i \frac{\partial}{\partial y^k} \otimes dx^i + \lambda \overset{\smile}{B}{}^k_i \frac{\delta}{\delta x^k} \otimes \delta y^i \tag{1.10}$$

where:

$$A^k_i = -\frac{a}{c}\delta^k_i + \frac{ad+bc}{dcF^2} y_i y^k, \quad \overset{\smile}{A}{}^k_i = -\frac{a}{c}\delta^k_i \tag{1.11}$$

$$B^k_i = \frac{c}{a}\delta^k_i - \frac{ad+bc}{abF^2} y_i y^k, \quad \overset{\smile}{B}{}^k_i = \frac{c}{a}\delta^k_i \tag{1.12}$$

Definition 1.2. *a) A linear connection on TM is said to be compatible with the almost 2–π structure Φ if it satisfies the condition:*

$$D_X \Phi = 0, \quad \forall X \in \chi(TM) \tag{1.13}$$

b) A linear connection on TM is said to be compatible with the metrical almost 2–π structure (G, Φ), *if it satisfies the conditions:*

$$D_X G = 0, \quad D_X \Phi = 0, \quad \forall X \in \chi(TM) \tag{1.14}$$

After some calculations, one obtains the following result:

Theorem 1.1. *The set of all d–connections compatible with the metrical almost 2–π structure* $(G_{a,a,c,c}, \Phi_{a,a,c,c})$ *is given by the following local coefficients:*

$$\overset{(H)}{F}{}^i_{jk} = \gamma^i_{jk} + \Omega^{ei}_{jm} \cdot X^m_{ek} \tag{1.15}$$

$$\overset{(V)}{F}{}^i_{jk} = \gamma^i_{jk} + \Omega^{ei}_{jm} \cdot X^m_{ek} \tag{1.16}$$

$$\overset{(H)}{C}{}^i_{jk} = p \cdot \delta^i_j \cdot y_k + \Omega^{ei}_{jm} \cdot U^m_{ek} \tag{1.17}$$

$$\overset{(V)}{C}{}^i_{jk} = q \cdot \delta^i_j \cdot y_k + \Omega^{ei}_{jm} \cdot U^m_{ek} \tag{1.18}$$

with $p = \dfrac{2a'F^2 - a}{aF^2}$ *and* $q = \dfrac{2c'F^2 - c}{cF^2}$, X^m_{ek} , U^m_{ek} *arbitrary d–tensor fields, and* Ω^{ei}_{jm} *the Obata operator of the Riemannian structure* γ.

Particular cases:

1^0. In the case $X^m_{ek} = U^m_{ek} = 0$ one obtains a d–connection compatible with the metrical almost 2–π structure $(G_{a,a,c,c}, \Phi_{a,a,c,c})$, which depends only on the Riemannian structure γ and the functions a, c. The local coefficients of this d–connection are as follows:

$$\overset{(H)}{F}{}^i_{jk} = \overset{(V)}{F}{}^i_{jk} = \gamma^i_{jk}, \quad \overset{(H)}{C}{}^i_{jk} = p \cdot \delta^i_j \cdot y_k, \quad \overset{(V)}{C}{}^i_{jk} = q \cdot \delta^i_j \cdot y_k \tag{1.19}$$

The simplicity of this d–connection and the fact that it is determined only by the Riemannian structure γ and by the functions a, c allows to call it the canonical d–connection of the space $(\widetilde{TM}, G_{a,a,c,c}, \Phi_{a,a,c,c})$.

2^0. If $a = F$, $c = k$, $k \in R^*$, one obtains the so called homogeneous metrical almost

2–π structure $(\overset{(0)}{G}, \overset{(0)}{\Phi})$ where the metrical structure $\overset{(0)}{G}$ is the Miron metrical structure from (1.8) and the almost 2–π structure $\overset{(0)}{\Phi}$ is defined by:

$$\overset{(0)}{\Phi} = -\lambda \cdot \frac{F}{k} \cdot \frac{\partial}{\partial y^i} \otimes dx^i + \lambda \cdot \frac{k}{F} \cdot \frac{\delta}{\delta x^i} \otimes \delta y^i \tag{1.20}$$

The canonical d-connection of the space $(\widetilde{TM}, \overset{(0)}{G}, \overset{(0)}{\Phi})$ is given as follows:

$$\overset{(H)}{F}{}^i_{jk} = \overset{(V)}{F}{}^i_{jk} = \gamma^i_{jk}, \quad \overset{(H)}{C}{}^i_{jk} = 0, \quad \overset{(V)}{C}{}^i_{jk} = -\frac{1}{F^2} \cdot \delta^i_j \cdot y_k \tag{1.21}$$

Remarks: The space $\overset{(2-\pi)}{M}{}^{2n} = (\widetilde{TM}, G_{a,a,c,c}, \Phi_{a,a,c,c})$ is called the (a,c)–geometrical model of the Riemannian space (M,γ) with respect to the metrical almost 2–π structure $(G_{a,a,c,c}, \Phi_{a,a,c,c})$ and $\overset{(0)}{M}{}^{2n} = (\widetilde{TM}, \overset{(0)}{G}, \overset{(0)}{\Phi})$ is called the homogeneous geometrical model of the Riemannian space (M,γ) with respect to the metrical almost 2–π structure $(\overset{(0)}{G}, \overset{(0)}{\Phi})$.

On $\widetilde{TM}$ there exists two remarkable vector fields $C = y^i \cdot \frac{\partial}{\partial y^i}$, called the Liouville vector field and $S = y^i \cdot \frac{\delta}{\delta x^i}$, which is the geodesic spray of R^n. A framed f–structure is a natural generalization of an almost contact structure. We recall its definition following [2]. Let N be a $(2n+s)-$ dimensional manifold endowed with an endomorphism f of rank $2n$, of the tangent bundle, satisfying $f^3 + f = 0$. If there exist on N the vector fields (ξ_α) and the 1–forms (η^α) $(\alpha = 1,2,\ldots,s)$ such that $\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta$, $f(\xi_\alpha) = 0$, $\eta^\alpha \circ f = 0$, $f^2 = -I + \sum_\alpha \eta^\alpha \otimes \xi_\alpha$, where I is the identity automorphism of the tangent bundle, then N is said to be a framed f–manifold.

2 A framed f – structure on tangent bundle

In the following we refer to $(G_{a,b,c,d}, \Phi_{a,b,c,d})$ metrical almost 2–π structure. We denote $\xi_1 = \frac{y^i}{b} \cdot \frac{\delta}{\delta x^i} = \frac{1}{b} \cdot S$ and $\xi_2 = \frac{y^i}{d} \cdot \frac{\partial}{\partial y^i} = \frac{1}{d} \cdot C$. We introduce the 1–forms $\eta^1 = \frac{by_i}{F^2} \cdot dx^i$ and $\eta^2 = \frac{dy_i}{F^2} \cdot \delta y^i$ which are globally defined on $\widetilde{TM}$. By a direct calculation one gets:

Proposition 2.1.
a) $\Phi_{a,b,c,d}(\xi_1) = -\xi_2$, $\Phi_{a,b,c,d}(\xi_2) = \xi_1$
b) $\eta^1 \circ \Phi_{a,b,c,d} = \eta^2$, $\eta^2 \circ \Phi_{a,b,c,d} = -\eta^1$
c) $\eta^1(X) = G_{a,b,c,d}(X, \xi_1)$, $\eta^2(X) = G_{a,b,c,d}(X, \xi_2)$, *for every* $X \in \mathcal{X}(\widetilde{TM})$.

Next we define a tensor field of type (1,1) on $\widetilde{TM}$ by:

$$f(X) = \Phi_{a,b,c,d} + \eta^1(X)\xi_2 - \eta^2(X)\xi_1, \quad X \in \mathcal{X}(\widetilde{TM}) \tag{2.1}$$

Proposition 2.2. *The 5–lot* $(f, \xi_1, \xi_2, \eta^1, \eta^2)$ *provides a framed f-structure on* $\widetilde{TM}$, *that is the following hold:*

a) $f^3 + f = 0$, $\text{rank}(f) = 2n - 2$;

b) $\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta$, $f(\xi_\alpha) = 0$, $\eta^\alpha \circ f = 0$;

c) $f^2(X) = -X + \eta^1(X)\xi_1 + \eta^2(X)\xi_2$, $X \in \mathcal{X}\left(\widetilde{TM}\right)$.

Proposition 2.3. *The Riemannian metric* $G_{a,b,c,d}$ *verifies:*

$$G_{a,b,c,d}(fX, fY) = G_{a,b,c,d}(X, Y) + 3\eta^1(X)\eta^1(Y) + 3\eta^2(X)\eta^2(Y) \tag{2.2}$$

for every $X, Y \in \mathcal{X}\left(\widetilde{TM}\right)$.

Proof. We mention only that:

$$f\left(\frac{\delta}{\delta x^p}\right) = \theta^k_p \cdot \frac{\partial}{\partial y^k}, \quad f\left(\frac{\partial}{\partial y^p}\right) = \Lambda^k_p \cdot \frac{\delta}{\delta x^p} \tag{2.3}$$

where:

$$\begin{cases} \theta^k_p = -\dfrac{a}{c}\delta^k_p + \dfrac{ad + 2bc}{cdF^2} y_p y^k \\ \Lambda^k_p = \dfrac{c}{a}\delta^k_p - \dfrac{2ad + bc}{abF^2} y_p y^k \end{cases} \tag{2.4}$$

Now, by a direct calculation, we can see that the relation (2.2) holds. q.e.d.

The operator f appears as a deformation of almost 2-π structures Φ similar with those studied in [2], [3] and [11]. Let us set $\Lambda(X, Y) = G_{a,b,c,d}(fX, Y)$ for every $X, Y \in \mathcal{X}\left(\widetilde{TM}\right)$. Using the above results we have $\Lambda(X, Y) = -\Lambda(Y, X)$ for every $X, Y \in \mathcal{X}\left(\widetilde{TM}\right)$. Thus Λ is a 2–form on $\widetilde{TM}$.

3 d–connections compatible with f

Definition 3.1. *A linear connection on* $\widetilde{TM}$ *is said to be compatible with the framed f-structure* f *if it satisfies the condition:*

$$D_X f = 0, \quad \forall X \in \chi(\widetilde{TM}) \tag{3.1}$$

In the adapted basis, any linear connection D on $\widetilde{TM}$ can be represented in the following way:

$$D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = \overset{(H)}{F}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(1)}{F}{}^i_{jk} \frac{\partial}{\partial y^i} \tag{3.2}$$

$$D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} = \overset{(2)}{F}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(V)}{F}{}^i_{jk} \frac{\partial}{\partial y^i} \tag{3.3}$$

$$D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} = \overset{(H)}{C}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(1)}{C}{}^i_{jk} \frac{\partial}{\partial y^i} \tag{3.4}$$

$$D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = \overset{(2)}{C}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(V)}{C}{}^i_{jk} \frac{\partial}{\partial y^i} \tag{3.5}$$

In the local coordinates, the condition (3.1) is equivalent with the following set of relations:

$$\frac{\delta \theta^p_m}{\delta x^k} - \theta^p_q \cdot \overset{(H)}{F}{}^q_{mk} + \theta^i_m \cdot \overset{(V)}{F}{}^p_{ik} = 0 \tag{3.6}$$

$$\theta^i_m \cdot \overset{(2)}{F}{}^n_{ik} - \Lambda^n_q \cdot \overset{(1)}{F}{}^q_{mk} = 0 \tag{3.7}$$

$$\frac{\delta \Lambda^i_q}{\delta x^k} + \Lambda^p_q \cdot \overset{(H)}{F}{}^i_{pk} - \Lambda^i_m \cdot \overset{(V)}{F}{}^m_{qk} = 0 \tag{3.8}$$

$$-\theta^n_q \cdot \overset{(2)}{F}{}^q_{mk} + \Lambda^p_m \cdot \overset{(1)}{F}{}^n_{pk} = 0 \tag{3.9}$$

$$\frac{\partial \theta^p_q}{\partial y^k} - \theta^p_m \cdot \overset{(H)}{C}{}^m_{qk} + \theta^m_q \cdot \overset{(V)}{C}{}^p_{mk} = 0 \tag{3.10}$$

$$\theta^m_q \cdot \overset{(2)}{C}{}^p_{mk} - \Lambda^p_m \cdot \overset{(1)}{C}{}^m_{qk} = 0 \tag{3.11}$$

$$\frac{\partial \Lambda^p_q}{\partial y^k} + \Lambda^m_q \cdot \overset{(H)}{C}{}^p_{mk} - \Lambda^p_m \cdot \overset{(V)}{C}{}^m_{qk} = 0 \tag{3.12}$$

$$-\Lambda^m_q \cdot \overset{(1)}{C}{}^p_{mk} + \theta^p_m \cdot \overset{(2)}{C}{}^m_{qk} = 0 \tag{3.13}$$

If we investigate only d–connections compatible with the framed f-structure f, then it is necessary to take into consideration (3.6), (3.8), (3.10) and (3.12) only.

Next, we shall give two results with respect to the existence of d–connections compatible with the framed f–structure f. For the beginning we make the following notations:

$$A = \frac{ad + 2bc}{2abF^2}, \quad B = \frac{2ad + bc}{2cdF^2}, \quad e = \frac{bc}{ad} \tag{3.14}$$

$$\begin{cases} \overline{\theta}^i_j = -\frac{c}{a}\delta^i_j + Ay_j y^i \\ \overline{\Lambda}^i_j = \frac{a}{c}\delta^i_j - By_j y^i \end{cases} \tag{3.15}$$

$$\overline{\gamma}^p_{mk} = -\frac{\delta (y_m \cdot y^p)}{\delta x^k} + \gamma^q_{mk} \cdot y_q \cdot y^p - \gamma^p_{qk} \cdot y_m \cdot y^q \tag{3.16}$$

$$Y^i_{jk} = \frac{\delta \theta^i_p}{\delta x^k} \cdot \overline{\theta}^p_j - \frac{\delta \Lambda^p_j}{\delta x^k} \cdot \overline{\Lambda}^i_p \tag{3.17}$$

$$Z^{in}_{mj} = -\frac{3e}{2F^2} \cdot y^i \cdot \delta^n_j \cdot y_m + \frac{3}{2eF^2} \cdot \delta^i_m \cdot y^n \cdot y_j + \frac{3\left(e^2 - 1\right)}{2eF^4} \cdot y^i \cdot y^n \cdot y_m \cdot y_j \tag{3.18}$$

Theorem 3.1. *If $A \cdot \Lambda^i_m \cdot \overline{\gamma}^m_{jk} = B \cdot \theta^m_j \cdot \overline{\gamma}^i_{mk}$, then there are d–connections on $\widetilde{TM}$ compatible with the framed f–structure f. One of them has the following coefficients:*

$$\overset{(H,1)}{F}{}^{i}_{\ jk} = \gamma^i_{jk} \tag{3.19}$$

$$\overset{(V,1)}{F}{}^{i}_{\ jk} = \gamma^i_{jk} + A \cdot \overline{\gamma}^i_{mk} \cdot \overline{\theta}^m_j \tag{3.20}$$

$$\overset{(H)}{C}{}^{i}_{\ jk} = 2 \cdot \left(\ln\left(\frac{a}{c}\right)\right)' \cdot \delta^i_j \cdot y_k - \frac{1}{F^2} \cdot \delta^i_k \cdot y_j + \frac{1}{F^2} \cdot \left(e^2\right)' \cdot y^i \cdot y_j \cdot y_k + \frac{1}{F^2} \cdot \gamma_{jk} \cdot y^i \tag{3.21}$$

$$\overset{(V)}{C}{}^{i}_{\ jk} = -\frac{1}{F^2} \cdot \delta^i_k \cdot y_j + \frac{1}{F^2} \cdot \gamma_{jk} \cdot y^i \tag{3.22}$$

Theorem 3.2. *If there exists $\overline{Z}^{jr}_{si}$ so that $Z^{in}_{mj} \cdot \overline{Z}^{jr}_{si} = \delta^n_s \cdot \delta^r_m$, then there are d–connections on $\widetilde{TM}$ compatible with the framed f–structure f. One of them has its coefficients given by:*

$$\overset{(H,2)}{F}{}^{i}_{\ jk} = Y^m_{nk} \cdot \overline{Z}^{ni}_{jm} \tag{3.23}$$

$$\overset{(V,2)}{F}{}^{i}_{\ jk} = Y^r_{sk} \cdot \overline{Z}^{sm}_{pr} \cdot \theta^i_m \cdot \overline{\theta}^p_j - \frac{\delta\theta^i_p}{\delta x^k} \cdot \overline{\theta}^p_j \tag{3.24}$$

and by relations (3.21) and (3.22).

Remark. In the same manner, we can study the framed f–structure determined by almost 2–π structure $\Phi_{a,a,c,c}$. For this framed f–structure we have the following properties:

a) $\Phi_{a,a,c,c}(\xi_1) = -\xi_2$, $\Phi_{a,a,c,c}(\xi_2) = \xi_1$

b) $\eta^1 \circ \Phi_{a,a,c,c} = \eta^2$, $\eta^2 \circ \Phi_{a,a,c,c} = -\eta^1$

c) $\eta^1(X) = G_{a,a,c,c}(X, \xi_1)$, $\eta^2(X) = G_{a,a,c,c}(X, \xi_2)$, for every $X \in \mathcal{X}\left(\widetilde{TM}\right)$.

d) $f^3 + f = 0$, $\text{rank}(f) = 2n - 2$;

e) $\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta$, $f(\xi_\alpha) = 0$, $\eta^\alpha \circ f = 0$;

f) $f^2(X) = -X + \eta^1(X)\xi_1 + \eta^2(X)\xi_2$, $X \in \mathcal{X}\left(\widetilde{TM}\right)$.

g) The Riemannian metric $G_{a,b,c,d}$ verifies:

$$G_{a,a,c,c}(fX, fY) = G_{a,a,c,c}(X,Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y)$$

for every $X, Y \in X\left(\widetilde{TM}\right)$, where

$$\xi_1 = \frac{y^i}{a} \cdot \frac{\delta}{\delta x^i} = \frac{1}{a} \cdot S, \quad \xi_2 = \frac{y^i}{c} \cdot \frac{\partial}{\partial y^i} = \frac{1}{c} \cdot C, \quad \eta^1 = \frac{a y_i}{F^2} \cdot dx^i, \quad \eta^2 = \frac{c y_i}{F^2} \cdot \delta y^i$$

and $f(X) = \Phi_{a,a,c,c} + \eta^1(X)\xi_2 - \eta^2(X)\xi_1$, $X \in \mathcal{X}\left(\widetilde{TM}\right)$

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