

Noether Symmetries for 1D Spinning Particle

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Abstract. In this paper we obtain five Noether symmetries and corresponding Noetherian first integrals for one-dimensional spinning particle. Only the first from these Noetherian conservation quantities was previously known.

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The aim of present paper is to obtain some Noether symmetries for classical spinning particle. To this purpose, let us recall the theory of Noether symmetries for higher-order Lagrangians.

In our study we treat only one dependent variable. This is for purposes of clarity because the case of several dependent variables is obtained by the inclusion of appropriately positioned indices. The general results will simple be stated. An interesting discussion appears for the 2D case; see [2].

Let $L = L(t, q^{(0)}, q^{(1)}, \dots, q^{(k)})$ be a Lagrangian of order k with usual action integral $A = \int L dt$ where $q^{(0)} = q$, $q^{(1)} = \frac{dq}{dt}$, \dots , $q^{(k)} = \frac{d^k q}{dt^k}$. Let $S = \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q}$ be a given vector field which yields the infinitesimal transformation (flow):

$$\tilde{t} = t + \varepsilon \tau, \quad \tilde{q} = q + \varepsilon \eta. \quad (1)$$

The new Lagrangian will be: $\tilde{L} = L\left(\tilde{t}, \tilde{q}, \frac{d\tilde{q}}{d\tilde{t}}, \dots, \frac{d^k \tilde{q}}{d\tilde{t}^k}\right)$ and the corresponding action $\tilde{A} = \int \tilde{L} d\tilde{t}$.

In the following, we adopt the approach of [8] (see also [5]); another way to obtain a Noether-type theorem in higher-order mechanics, based on higher-order energies, appears in [7] for the autonomous case (i.e. independent of time Lagrangians) and in [1] for the time-dependent case; also, for a non-Noetherian way to obtain first integrals for higher-order differential equations see [3]. Namely, S is called a *Noether symmetry* if the action is invariant by the flow (1) of S i.e. $\tilde{A} = A$. This is equivalent with the existence of a function f , usually called *gauge* (or *boundary term* in Noether's original work), such that the following *Killing equation* holds ([8, p. 224-225]):

$$\frac{df}{dt} = \frac{d\tau}{dt} L + \tau \frac{\partial L}{\partial t} + \sum_{i=0}^k \xi_i \frac{\partial L}{\partial q^{(i)}} \quad (2)$$

where:

$$\begin{cases} \xi_0 = \eta \\ \xi_1 = \frac{d\xi_0}{dt} - q^{(1)} \frac{d\tau}{dt} \\ \dots \\ \xi_i = \frac{d\xi_{i-1}}{dt} - q^{(i)} \frac{d\tau}{dt} \end{cases} \quad (3)$$

In this case the function:

$$\mathcal{F} = f - \left[\tau L + \sum_{j=0}^{k-1} \sum_{i=j+1}^k (-1)^j (\xi_{i-j-1} - \tau q^{(i-j)}) \frac{d^j}{dt^j} \left(\frac{\partial L}{\partial q^{(i)}} \right) \right] \quad (4)$$

is a Noetherian first integral.

Let us treat in detail the particular case $k = 2$. The Killing equation becomes:

$$\frac{df}{dt} = \frac{d\tau}{dt} L + \tau \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \left(\frac{d\eta}{dt} - q^{(1)} \frac{d\tau}{dt} \right) \frac{\partial L}{\partial q^{(1)}} + \left(\frac{d^2\eta}{dt^2} - q^{(1)} \frac{d^2\tau}{dt^2} - 2q^{(2)} \frac{d\tau}{dt} \right) \frac{\partial L}{\partial q^{(2)}} \quad (5)$$

and the Noetherian first integral is:

$$\mathcal{F} = f - \tau L - \left(\eta - q^{(1)}\tau \right) \left[\frac{\partial L}{\partial q^{(1)}} - \frac{d}{dt} \left(\frac{\partial L}{\partial q^{(2)}} \right) \right] - \left[\frac{d\eta}{dt} - q^{(1)} \frac{d\tau}{dt} - q^{(2)}\tau \right] \frac{\partial L}{\partial q^{(2)}}. \quad (6)$$

Example: the spinning particle

After [6, p.41-47], the Lagrangian of classical spinning particle is of second order, namely:

$$L = \frac{1}{2} \left(q^{(1)} \right)^2 - \frac{1}{2} \left(q^{(2)} \right)^2 \quad (7)$$

with the Euler-Lagrange equation:

$$q^{(4)} + q^{(2)} = 0. \quad (8)$$

(For a detailed discussion with historical arguments see [13]). The two-dimensional case is also very interesting from the point of view of Noetherian symmetries, cf. [2]. The Killing equation (5) is:

$$\frac{df}{dt} = \frac{d\tau}{dt} L + q^{(1)} \left(\frac{d\eta}{dt} - q^{(1)} \frac{d\tau}{dt} \right) - q^{(2)} \left(\frac{d^2\eta}{dt^2} - q^{(1)} \frac{d^2\tau}{dt^2} - 2q^{(2)} \frac{d\tau}{dt} \right) \quad (9)$$

and the Noetherian first integral (6) is:

$$\mathcal{F} = f - \tau L - \left(\eta - q^{(1)}\tau \right) \left(q + q^{(3)} \right) + \left[\frac{d\eta}{dt} - q^{(1)} \frac{d\tau}{dt} - q^{(2)}\tau \right] q^{(2)}. \quad (10)$$

Looking at some calculations for harmonic oscillator from [4] we obtain the following Noether symmetries:

$$(i) \quad S_1 = \frac{\partial}{\partial t} \quad (11)$$

with the gauge $G_1 = 0$ and the Noetherian first integral:

$$\mathcal{F}_1 = \frac{1}{2} \left(q^{(1)} \right)^2 - \frac{1}{2} \left(q^{(2)} \right)^2 + q^{(1)} q^{(3)} \quad (12)$$

which is exactly the energy (or Hamiltonian) of the Lagrangian (7), cf. [6, p.41-48].

$$(ii) \quad S_2 = \cos t \frac{\partial}{\partial q} \quad (13)$$

with the gauge $G_2 = \cos tq^{(1)}$ and the Noetherian first integral:

$$\mathcal{F}_2 = q^{(2)} \sin t + q^{(3)} \cos t. \quad (14)$$

$$(iii) \quad S_3 = \sin t \frac{\partial}{\partial q} \quad (15)$$

with the gauge $G_3 = \sin tq^{(1)}$ and the Noetherian first integral:

$$\mathcal{F}_3 = q^{(2)} \cos t - q^{(3)} \sin t. \quad (16)$$

$$(iv) \quad S_4 = \sin 2t \frac{\partial}{\partial t} + q^{(1)} \cos 2t \frac{\partial}{\partial q} \quad (17)$$

with the gauge:

$$G_4 = \sin 2t \left((q^{(2)})^2 - \frac{1}{2} (q^{(1)})^2 - \frac{1}{2} (q^{(3)})^2 + 2q^{(1)}q^{(2)} - q^{(1)}q^{(3)} \right) + \\ + \cos 2t \left((q^{(1)})^2 - (q^{(2)})^2 + 2q^{(1)}q^{(2)} + q^{(1)}q^{(3)} + q^{(2)}q^{(3)} \right) \quad (18)$$

and the Noetherian first integral:

$$\mathcal{F}_4 = \frac{1}{2} \left((q^{(2)})^2 - (q^{(3)})^2 \right) \sin 2t + q^{(2)}q^{(3)} \cos 2t \quad (19)$$

$$(v) \quad S_5 = \cos 2t \frac{\partial}{\partial t} - q^{(1)} \sin 2t \frac{\partial}{\partial q} \quad (20)$$

with the gauge:

$$G_5 = \cos 2t \left((q^{(2)})^2 - \frac{1}{2} (q^{(1)})^2 - \frac{1}{2} (q^{(3)})^2 + 2q^{(1)}q^{(2)} - q^{(1)}q^{(3)} \right) + \\ + \sin 2t \left(- (q^{(1)})^2 + (q^{(2)})^2 - 2q^{(1)}q^{(2)} - q^{(1)}q^{(3)} - q^{(2)}q^{(3)} \right) \quad (21)$$

and the Noetherian first integral:

$$\mathcal{F}_5 = \frac{1}{2} \left((q^{(2)})^2 - (q^{(3)})^2 \right) \cos 2t - q^{(2)}q^{(3)} \sin 2t. \quad (22)$$

Let us point a common expression for last two cases. Namely, setting:

$$\begin{cases} A = A(q^{(1)}, q^{(2)}, q^{(3)}) = (q^{(1)})^2 - (q^{(2)})^2 + 2q^{(1)}q^{(2)} + q^{(1)}q^{(3)} + q^{(2)}q^{(3)} \\ B = B(q^{(1)}, q^{(2)}, q^{(3)}) = (q^{(2)})^2 - \frac{1}{2} (q^{(1)})^2 - \frac{1}{2} (q^{(3)})^2 + 2q^{(1)}q^{(3)} \end{cases} \quad (23)$$

it results:

$$G_4 = A \cos 2t + B \sin 2t, \quad G_5 = B \cos 2t - A \sin 2t \quad (24)$$

and denoting:

$$C = C(q^{(2)}, q^{(3)}) = q^{(2)}q^{(3)}, \quad D = D(q^{(2)}, q^{(3)}) = \frac{1}{2} \left((q^{(2)})^2 - (q^{(3)})^2 \right) \quad (25)$$

we get:

$$\mathcal{F}_4 = C \cos 2t + D \sin 2t, \quad \mathcal{F}_5 = D \cos 2t - C \sin 2t. \quad (26)$$

Also, the relations (14) - (16), (24), (26) have the matrix form:

$$\begin{pmatrix} \mathcal{F}_2 \\ \mathcal{F}_3 \end{pmatrix} = \begin{pmatrix} q^{(3)} \\ q^{(2)} \end{pmatrix} \cdot \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad (14') - (16')$$

$$\begin{pmatrix} G_4 \\ G_5 \end{pmatrix} = (A, B) \cdot \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \quad (24')$$

$$\begin{pmatrix} \mathcal{F}_4 \\ \mathcal{F}_5 \end{pmatrix} = (C, D) \cdot \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \quad (26')$$

and it's amazing the "birth" of Lie group $SO(2) \simeq S^1$ through the trigonometric matrix from the RHS of these equations!

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