

## Generalized Lagrange Spaces Derived from a Finsler - Synge Metric

Valer NIMINET

**Abstract.** The Finsler- Synge metric may be thought as a deformation of a Riemannian or a Finslerian metric as in [1]. Such a metric does not reduce to a Lagrangian one and in fact it belongs to the class of generalized Lagrange metrics as these were presented and studied in the book [2] by R. Miron and M. Anastasiei. The geometry of this generalized Lagrange metric is influenced by the Finsler space that it deforms. We study mainly the case when the said Finsler space is locally Minkowski.

### 1 Introduction

In a recent paper, [4], Singh U. P. studied the generalized Lagrange spaces, shortly *GL*-spaces,  $GL^n = (M, g_{ij}(x, y))$  with the fundamental tensor

$$g_{ij}(x, y) = \gamma_{ij}(x, y) + (1/c^2)y_i y_j, \quad y_i = \gamma_{ij}(x, y)y^j, \quad (1.1)$$

where  $M$  is a smooth manifold and  $\gamma_{ij}(x, y)$  is the fundamental tensor of a Finsler space  $F^n = (M, F(x, y))$ .

He investigated these spaces in the cases when  $F^n$  is a locally Minkowski, Landsberg or Berwald Finsler space.

The case when  $\gamma_{ij}(x, y)$  do not depend on the directional variable, that is when the *GL*-metric ( $g_{ij}$ ) from (1.1) reduces to

$$g_{ij}(x, y) = \gamma_{ij}(x) + (1/c^2)y_i y_j, \quad y_i = \gamma_{ij}(x)y^j, \quad (1.2)$$

was studied by R. Miron and T. Kawaguchi, [3].

The same authors consider in the excellent paper, [3], the *GL*-metric

$$g_{ij}(x, y) = \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, y)}\right) y_i y_j, \quad y_i = \gamma_{ij}(x)y^j, \quad (1.3)$$

where the function  $n > 1$  is the so-called refractive index. It is obvious that for

$$\frac{1}{n^2} = 1 - \frac{1}{c^2}, \quad (1.4)$$

the *GL*-metric (1.3) coincides with the *GL*-metric (1.2).

Let  $V^i(x)$  be a local vector field on  $M$  and  $S_V : M \rightarrow TM$  be the local section,

The restriction of the tensor field  $g_{ij}$  from (1.3) to the image of the mapping  $S_V$  gives the Synge metric from the Relativistic Optics:

$$g_{ij}(x, V(x)) = \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, V(x))}\right) V_i V_j, \quad V_i(x) = \gamma_{ij}(x) V^j(x). \quad (1.5)$$

This fact shows the importance of the  $GL$ -metric (1.3) for Relativistic Optics.

On the other hand, R. Miron and M. Anastasiei extended (see Ch. XII of the book [2]) the  $GL$ -metric (1.3) to a Finsler - Synge metric

$$g_{ij}(x, y) = \gamma_{ij}(x, y) + \left(1 - \frac{1}{n^2(x, y)}\right) y_i y_j, \quad y_i = \gamma_{ij}(x, y) y^j, \quad (1.6)$$

where  $\gamma_{ij}(x, y)$  is the fundamental tensor of a Finsler space  $F^n = (M, F(x, y))$ . This reduces to the  $GL$ -metric (1.1) if the condition (1.4) holds.

In this paper we extend the results from Singh's paper [4] by studying the  $GL$ -metric (1.6) when the Finsler space  $F^n$  is a locally Minkowski space. Moreover, in the last section we replace the nonlinear connection with a new one and discuss again the case when  $F^n$  is a locally Minkowski space.

## 2 Preliminaries

We recall following the book [2] by R. Miron and M. Anastasiei some facts regarding the  $GL$ -metric (1.6).

First, we have

$$y_i = \gamma_{ij}(x, y) y^j = \frac{1}{2} \frac{\partial F^2}{\partial y^i}, \quad (2.1)$$

$$\|y\|^2 := y_i y^i = F^2. \quad (2.2)$$

The contravariant tensor field  $g^{ij}(x, y)$  has the form

$$g^{ij}(x, y) = \gamma^{ij}(x, y) - \frac{1}{a(x, y)} \left(1 - \frac{1}{n^2(x, y)}\right) y^i y^j, \quad (2.3)$$

where

$$a(x, y) = 1 + \left(1 - \frac{1}{n^2}\right) F^2 > 1. \quad (2.4)$$

The  $GL$ -metric (1.6) is not reducible to a Lagrange or a Finsler metric.

We postulate, as R. Miron and M. Anastasiei have done, that the  $GL$ -space  $GL^n = (M, g_{ij}(x, y))$  with the  $GL$ -metric  $g_{ij}(x, y)$  given by (1.6) is equipped with the Cartan nonlinear connection  $N$  of the associated Finsler space  $F^n$ . Its local coefficients  $N_j^i(x, y)$  are given by

$$N_j^i(x, y) = \frac{1}{2} \frac{\partial \gamma_{00}^i}{\partial y^j}, \quad (2.5)$$

where  $\gamma_{00}^i = \gamma_{jk}^i(x, y) y^j y^k$  and  $\gamma_{jk}^i(x, y)$  are the Christoffel symbols constructed using  $\gamma_{ij}(x, y)$ .

The autoparallel curves of  $N$  coincide with the geodesics of  $F^n$ .

Let us consider the basis adapted to the horizontal distribution  $N$  and to the vertical

distribution  $V$  on  $T_0M := TM \setminus 0$ .

This is  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ ,  $i = \overline{1, n}$  with

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y) \frac{\partial}{\partial y^j}. \quad (2.6)$$

The dual of this adapted basis is  $(dx^i, \delta y^i)$ ,  $i = \overline{1, n}$  with

$$\delta y^i = dy^i + N_j^i(x, y) dx^j. \quad (2.7)$$

The geometrical objects

$$G = g_{ij} dx^i \otimes dx^j + g_{ij}(x, y) \delta y^i \otimes \delta y^j, \quad (2.7')$$

$$F = -\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^i, \quad (2.7'')$$

determine an almost Hermitian structure  $(G, F)$  on  $T_0M$ .

It depends on  $\gamma_{ij}(x)$  only, hence on the nature of the Finsler space  $F^n$ . The linear connection  $D$  on  $T_0M$  with the properties:  $DG = 0$ ,  $DF = 0$ ,  $(hh)h$ -torsion and  $(vv)v$ -torsion vanishing has the local coefficients

$$\begin{aligned} L_{jk}^i &= \frac{1}{2} g^{is} \left( \frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} \right), \\ C_{jk}^i &= \frac{1}{2} g^{is} \left( \frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{ik}}{\partial y^s} \right). \end{aligned} \quad (2.8)$$

This is the canonical metrical connection of the  $GL$ -space

$$GL^n = (M, g_{ij}(x, y)).$$

By Theorem 4.1, Ch. XII from [2], this connection has the following coefficients:

$$L_{jk}^i = \overset{0}{F}_{jk}^i + \overset{i}{\bigwedge}_{jk}, \quad C_{jk}^i = \overset{0}{C}_{jk}^i + \overset{1}{C}_{jk}^i, \quad (2.9)$$

where  $\overset{0}{C}\Gamma(N) = (\overset{0}{F}_{jk}^i, \overset{0}{C}_{jk}^i)$  is the Cartan linear connection of  $F^n$  and

$$\overset{i}{\bigwedge}_{jk} = g^{ih} \overset{i}{\bigwedge}_{jhk}, \quad \overset{1}{C}_{jk}^i = g^{ih} \overset{1}{C}_{jhk}^i, \quad (2.9')$$

with

$$\overset{i}{\bigwedge}_{ijk} = -u \left( y_i y_j \frac{\delta u}{\delta x^k} + y_j y_k \frac{\delta u}{\delta x^i} - y_k y_i \frac{\delta u}{\delta x^j} \right), \quad (2.10)$$

$$\begin{aligned} \overset{1}{C}_{ijk} &= -u \left( y_i y_j \frac{\partial u}{\partial y^k} + y_j y_k \frac{\partial u}{\partial y^i} - y_k y_i \frac{\partial u}{\partial y^j} \right), \\ u(x, y) &= \frac{1}{n(x, y)}. \end{aligned} \quad (2.10')$$

The above facts will be used for studying the  $GL$ -metric (1.6) for  $F^n$  a locally Minkowski space.

### 3 The case when $F^n$ is a locally Minkowski space

In this case around every point  $u = (x, y) \in T_0M := TM \setminus 0$  there exists a system of local coordinates in which the function  $F$  does not depend on  $(x^i)$ . In such a system of coordinates, the metric  $(\gamma_{ij})$  depends on  $(y^i)$  only, the Christoffel symbols vanish and so the coefficients of the nonlinear connection vanish, too. If, moreover, one assumes that the refractive index does not depend on  $(x^i)$ , that is  $\frac{\partial n}{\partial x^i} = 0 \forall i = \overline{1, n}$ , it results that the Finsler - Synge metric (1.6) has the property  $\frac{\partial g_{ij}}{\partial x^k} = 0$ .

Now we list some properties of the space  $GL^n = (M, g_{ij}(y))$  with

$$g_{ij}(y) = \gamma_{ij}(y) + \left(1 - \frac{1}{n^2(y)}\right) y_i y_j, \quad y_i = \gamma_{ij}(y) y^j. \quad (3.1)$$

#### Properties

1° The nonlinear connection  $N$  is integrable.

2° The autoparallel curves of  $N$  are solutions of the differential system  $\frac{d^2 x^i}{dt^2} = 0$ .

3° The refractive index is  $h$ -covariant constant, that is

$$\frac{\delta n}{\delta x^k} = 0. \quad (3.2)$$

4° If the optical medium is nondispersive, i.e.  $\frac{\partial n}{\partial y^i} = 0$ , then the refractive index is constant.

5° The canonical metrical connection  $CT(N) = (L_{jk}^i, C_{jk}^i)$  has the coefficients

$$L_{jk}^i = 0, \quad C_{jk}^i = \overset{0}{C}_{jk}^i + \overset{1}{C}_{jk}^i, \quad (3.3)$$

with  $\overset{0}{C}_{jk}^i = \frac{1}{4} \gamma^{ih} \frac{\partial^3 F^2}{\partial y^h \partial y^j \partial y^k}$  and  $\overset{1}{C}_{jk}^i$  from (2.9'), (2.10).

6° The  $h$ -paths of the space  $GL^n$  are solutions of the differential system  $\frac{d^2 x^i}{dt} = 0, \frac{dy^i}{dt} = 0$ .

7° The  $v$ -path of the space  $GL^n$  in a point  $x_0 \in M$  are characterized by

$$x^i = x_0^i, \quad \frac{d^2 y^i}{dt^2} + \left[ \overset{0}{C}_{jk}^i(y) + \overset{1}{C}_{jk}^i(y) \right] \frac{dy^j}{dt} \frac{dy^k}{dt} = 0.$$

8° The Einstein equations of the space  $GL^n$  reduce to

$$S_{ij} - \frac{1}{2} S g_{ij} = \kappa T_{ij} \text{ and } T_j^i|_i = 0.$$

We refer to Ch. X in [2] for the general form of these equations.

9° The  $h$ - and  $v$ -electromagnetic tensor field  $F_{ij}$  and  $f_{ij}$  are as follows:

$$F_{ij} = 0, \quad f_{ij} = -u(y) F^2(y) \left( y_j \frac{\partial u}{\partial y^i} - y_i \frac{\partial u}{\partial y^j} \right), \quad u(y) = \frac{1}{n(y)}$$

and the following generalized Maxwell equations hold:

$$f_{ij}|_k + f_{jk}|_i + f_{ki}|_j = 0.$$

For the general electromagnetic tensor fields we refer also to Ch. X in [2].

#### 4 The case of a new nonlinear connection $N^*$

Recall that we postulated that the  $GL$ -space with the Finsler - Synge metric (1.6) is equipped with the nonlinear connection  $N$  with the coefficients  $N_j^i$  given by (2.5). This nonlinear connection does not depend on the refractive index  $n(x, y)$ . In order to involve also the function  $n(x, y)$  at the nonlinear level we replace  $N$  with a nonlinear connection  $N^*$  of local coefficients

$$N_j^{*i} = N_j^i - a(x, y)\delta_j^i, \quad (4.1)$$

where  $a$  is the function introduced in (2.4).

Then we have

**Theorem 4.1.** *The autoparallel curves of  $N^*$  are solutions of the differential system*

$$\frac{dy^i}{dt} + \gamma_{jk}^i(x, y)y^j y^k = a(x, y)y^i, \quad y^i = \frac{dx^i}{dt}. \quad (4.2)$$

**Proof.** The autoparallel curves of  $N^*$  are solutions of the differential system

$$\frac{\delta^* y^i}{dt} := \frac{dy^i}{dt} + N_j^{*i} \frac{dx^j}{dt} = 0, \quad y^i = \frac{dx^i}{dt}.$$

By (4.1) and (2.5) this reduces to (4.2), q.e.d.

**Theorem 4.2.** *The metrical  $N^*$ -linear connection  $CT(N^*) = (L_{jk}^{*i}, C_{jk}^{*i})$  has the coefficients*

$$L_{jk}^{*i} = L_{jk}^i + aC_{jk}^i, \quad C_{jk}^{*i} = C_{jk}^i. \quad (4.3)$$

**Proof.** The said coefficients have the same form as in (2.8) with  $\frac{\delta}{\delta x^i}$  replaced with  $\frac{\delta^*}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^{*j} \frac{\partial}{\partial y^k}$ . The second formula in (4.3) is clear. The first follows by a direct computation using (4.1).

Now is again of interest the case  $1 - \frac{1}{n^2} = \frac{1}{c^2}$ .

Also, it is of interest to review the case when  $F^n$  is a locally Minkowski space now when  $N$  was replaced by  $N^*$ . We have (assuming that  $n$  depends on  $y$  only) the following

##### Properties

1°  $g_{ij}$  depends on  $(y^i)$  only,

2° the nonlinear connection has the coefficients  $N_j^{*i} = -a(y)\delta_j^i$ ,

3°  $N^*$  has the weakly torsion  $t_{jk}^{*i} := \frac{\partial N_j^{*i}}{\partial y^k} - \frac{\partial N_k^{*i}}{\partial y^j}$  of the form  $t_{jk}^{*i} = \delta_k^i \frac{\partial a}{\partial y^j} - \delta_j^i \frac{\partial a}{\partial y^k}$ ,

4° The curvature  $R_{jk}^{*i} := \frac{\delta^* N_j^{ki}}{\delta x^k} - \frac{\delta^k N_k^{*i}}{\delta x^j}$  of  $N^*$  takes the form  $R_{jk}^{*i} = -at_{jk}^{*i}$ ,

5°  $N^*$  is integrable if and only if  $t_{jk}^{*i} = 0$ ,

6°  $t_{jk}^{*i} = 0$  if and only if  $\frac{\partial a}{\partial y^i} = 0 \Leftrightarrow$

$$y^i \frac{\partial u^2}{\partial y^i} = 2(1 - u^2). \quad (4.4)$$

7° The autoparallel curves of  $N^*$  are solutions of the differential system

$$\frac{dy^i}{dt} - a(y)y^i = 0, \quad y^i = \frac{dx^i}{dt}. \quad (4.5)$$

8° The Berwald connection has the coefficients  $B_{jk}^i := \frac{\partial N_j^{*i}}{\partial y^k} = -\delta_j^i \frac{\partial a(y)}{\partial y^k}$ .

## References

- [1] Anastasiei, M., Shimada, S., *Deformations of Finsler Metrics*, In the volume **Finslerian Geometries** edited by P.L. Antonelli, Kluwer Academic Publishers, 2000.
- [2] Miron, R., Anastasiei, M., *The Geometry of Lagrange spaces: Theory and Applications* Kluwer Academic Publishers, FTPH 59, 1994.
- [3] Miron, R., Kawaguchi, T., *Relativistic Geometrical Optics*, Int. J. Theor. Phys. 30/11 (1991), 1521-1543.
- [4] Singh, U.P., *On the generalized Lagrange space and corresponding Lagrange space arising from the metric tensor  $g_{ij}(x, y) + (1/c^2)y_i y_j$* , to appear.