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On Semi-Symmetric Conformal Metrical d-Linear Connections

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Abstract. In the present paper starting from the notion of conformal metrical structure in the cotangent bundle, we define the notion of conformal metrical d-linear connection with respect to a conformal metrical structure corresponding to the 1-forms ω and $\tilde{\omega}$ in T^*M. We determine all conformal metrical d-linear connections in the case when the nonlinear connection is arbitrary and we find important particular cases. We find the transformation group of these connections. We study the role of the torsion tensor fields T and S in this theory, especially the semi-symmetric d-linear connections, and the group of transformations of semi-symmetric conformal metrical d-linear connections, which preserve the nonlinear connection N, and its important invariants.

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1 The notion of conformal metrical structure in the cotangent bundle.

The geometry of the cotangent bundle (T^*M, π^*, M) has been studied by R. Miron, S. Watanabe and S. Ikeda in [9], by K. Yano and S. Ishihara in [11], by Gh. Atanasiu and F. Klepp in [1], by R. Miron, D. Hrimiuc, H. Shimada and S. Sabău in [8] etc.

Concerning the terminology and notations, we use those from [7].

Let M be a real C^∞-differentiable manifold with dimension n, (T^*M, π^*, M) its cotangent bundle.

If (x^i) is a local coordinates system on a domain U of a chart on M, the induced system of coordinates on ${\pi^*}^{-1}(U)$ is (x^i, p_i), $(i = 1, ..., n)$.

Let N be a nonlinear connection on T^*M, with the coefficients $N_{ij}(x,p)$.

We consider on T^*M a metrical structure G:

$$G(x,p) = g_{ij}(x,p)dx^i \otimes dx^j + \tilde{g}^{ij}(x,p)\delta p_i \otimes \delta p_j, \tag{1.1}$$

where $(dx^i, \delta p_i)$, $(i = 1, ..., n)$ is the dual basis of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i}\right)$, and $(g_{ij}(x,p),\ \tilde{g}^{ij}(x,p))$ is a pair of given d-tensor fields on T^*M, of the types $(0,2)$, and $(2,0)$ respectively, each of them symmetric and nondegenerate.

We associate to the lift G the Obata operators:

$$\begin{cases} \Omega_{sj}^{ir} = \frac{1}{2}(\delta_s^i \delta_j^r - g_{sj} g^{ir}), & \Omega_{sj}^{*ir} = \frac{1}{2}(\delta_s^i \delta_j^r + g_{sj} g^{ir}), \\ \tilde{\Omega}_{sj}^{ir} = \frac{1}{2}(\delta_s^i \delta_j^r - \tilde{g}_{sj} \tilde{g}^{ir}), & \tilde{\Omega}_{sj}^{*ir} = \frac{1}{2}(\delta_s^i \delta_j^r + \tilde{g}_{sj} \tilde{g}^{ir}). \end{cases} \tag{1.2}$$

Obata's operators have the same properties as the ones associated with a Finsler space [7].

Let $\mathcal{S}_2(T^*M)$ be the set of all symmetric d-tensor fields, of the type $(0,2)$ or $(2,0)$ on T^*M. As is easily shown, the relation on $\mathcal{S}_2(T^*M)$ defined by (1.3):

$$\begin{cases} (a_{ij} \sim b_{ij}) \Leftrightarrow ((\exists)\, \lambda(x,p) \in \mathcal{F}(T^*M), \quad a_{ij}(x,p) = e^{2\lambda(x,p)} b_{ij}(x,p)), \\ \left(\tilde{a}_{ij} \sim \tilde{b}_{ij}\right) \Leftrightarrow \left((\exists)\, \mu(x,p) \in \mathcal{F}(T^*M), \quad \tilde{a}^{ij}(x,p) = e^{2\mu(x,p)} \tilde{b}^{ij}(x,p)\right), \end{cases} \tag{1.3}$$

is an equivalence relation on $\mathcal{S}_2(T^*M)$.

Definition 1.1. *The equivalence class: $\hat{G}$ of $\mathcal{S}_2(T^*M)/_\sim$ to which the metric tensor field G belongs, is called conformal metrical structure on T^*M.*

Thus:

$$\hat{G} = \{G' \mid G'_{ij}(x,p) = e^{2\lambda(x,p)} g_{ij}(x,p) \text{ and } G'^{\,ij}(x,p) = e^{2\mu(x,p)} \tilde{g}^{ij}(x,p)\}. \tag{1.4}$$

2 Conformal metrical d-linear connections.

Definition 2.1. *A d-linear connection, D^*, on T^*M, with local coefficients $D^*\Gamma(N) = (H^i_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^{\,i}{}_j{}^k, C_i{}^{jk})$, for which there exists the 1-forms ω and $\tilde{\omega}$ in T^*M, $\omega = \omega_i dx^i + \dot{\omega}^i \delta p_i$, $\tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}^i \delta p_i$ such that:*

$$\begin{cases} g_{ij|k} = 2\omega_k g_{ij}, \; g_{ij}|^k = 2\dot{\omega}^k g_{ij}, \\ \tilde{g}^{ij}{}_{|k} = 2\tilde{\omega}_k \tilde{g}^{ij}, \; \tilde{g}^{ij}|^k = 2\dot{\tilde{\omega}}^k \tilde{g}^{ij}, \end{cases} \tag{2.1}$$

where $|$ and $|$ denote the h- and v-covariant derivatives with respect to D^*, is called conformal metrical d-linear connection, with respect to the conformal metrical structure $\hat{G}$, corresponding to the 1-forms ω, $\tilde{\omega}$ and is denoted by: $D^*\Gamma(N, \omega, \tilde{\omega})$.

For any representative $G' \in \hat{G}$, $G' = g'_{ij}(x,p) dx^i \otimes dx^j + \tilde{g}'^{\,ij} \delta p_i \otimes \delta p_j$, we have:

Theorem 2.1. *For $g'_{ij} = e^{2\lambda} g_{ij}$ and $\tilde{g}'^{\,ij} = e^{2\mu} \tilde{g}^{ij}$, a conformal metrical d-linear connection with respect to the conformal metrical structure $\hat{G}$, corresponding to the 1-forms ω, $\tilde{\omega}$ in T^*M, $D^*\Gamma(N, \omega, \tilde{\omega})$ satisfies:*

$$\begin{cases} g'_{ij|k} = 2\omega'_k g'_{ij}, \; g'_{ij}|^k = 2\dot{\omega}'^{\,k} g'_{ij}, \\ \tilde{g}'^{\,ij}{}_{|k} = 2\tilde{\omega}'_k \tilde{g}^{ij\,\prime}, \tilde{g}'^{\,ij}|^k = 2\dot{\tilde{\omega}}'^{\,k} \tilde{g}^{ij\,\prime}, \end{cases} \tag{2.2}$$

where $\omega' = \omega + d\lambda$ and $\tilde{\omega}' = \tilde{\omega} + d\mu$.

Since in Theorem 2.1 $\omega' = \tilde{\omega}' = 0$ is equivalent to $\omega = d(-\lambda)$ and $\tilde{\omega} = d(-\mu)$, we have:

Theorem 2.2. *A conformal metrical d-linear connection, with respect to $\hat{G}$, corresponding to the 1-forms ω, $\tilde{\omega}$ in T^*M, denoted by: $D^*\Gamma(N,\omega,\tilde{\omega})$, is metrical with respect to $G' \in \hat{G}$, i.e. ${g'}_{ij|k} = {g'}_{ij}|^k = {\tilde{g}'}{}^{ij}{}_{|k} = {\tilde{g}'}{}^{ij}|^k = 0$ if and only if ω and $\tilde{\omega}$ are exact.*

We shall determine the set of all conformal metrical d-linear connections, with respect to $\hat{G}$.

Let $\overset{0}{D}{}^*\Gamma(\overset{0}{N}) = (\overset{0}{H}{}^i{}_{jm}, \overset{0}{\tilde{H}}{}^i{}_{jm}, \overset{0}{\tilde{C}}{}^i{}_j{}^k, \overset{0}{C}{}^i{}_{jk})$ be the local coefficients of a fixed d-linear connection on T^*M. Then any d-linear connection, D^*, on T^*M , can be expressed in the form:

$$\begin{cases} N_{ij} = \overset{0}{N}_{ij} - A_{ij}, \\ H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + A_{lk}\overset{0}{\tilde{C}}{}^i{}_j{}^l - B^i{}_{jk}, \\ \tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k + A_{lk}\overset{0}{C}{}_i{}^{jl} - \tilde{B}_i{}^j{}_k, \\ \tilde{C}_i{}^j{}_k = \overset{0}{\tilde{C}}{}^i{}_j{}^k - \tilde{D}^i{}_j{}^k, \\ C_i{}^{jk} = \overset{0}{C}{}_i{}^{jk} - D_i{}^{jk}, \\ A_{ij\overset{0}{|}k} = 0, \end{cases} \tag{2.3}$$

where $(A_{ij}, B^i{}_{jk}, \tilde{B}_i{}^j{}_k, \tilde{D}^i{}_j{}^k, D_i{}^{jk})$ are components of the difference tensor fields of $D^*\Gamma(N)$ from $\overset{0}{D}{}^*\Gamma(\overset{0}{N})$, [3].

Theorem 2.3. *Let $\overset{0}{D}{}^*$ be a given d-linear connection on T^*M, with local coefficients $\overset{0}{D}{}^*\Gamma(\overset{0}{N}) = (\overset{0}{H}{}^i{}_{jk}, \overset{0}{\tilde{H}}{}^i{}_{jk}, \overset{0}{\tilde{C}}{}^i{}_j{}^k, \overset{0}{C}{}_i{}^{jk})$. The set of all conformal metrical d-linear connections on T^*M, with respect to $\hat{G}$, corresponding to the 1-forms ω and $\tilde{\omega}$, with local coefficients $D^*\Gamma(N,\omega,\tilde{\omega}) = (H^i{}_{jk}, \tilde{H}^i{}_{jk}, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$ is given by:*

$$\begin{cases} N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\ H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + X_{lk}\overset{0}{\tilde{C}}{}^i{}_j{}^l + \frac{1}{2}g^{ir}(g_{rj\overset{0}{|}k} - 2\omega_k g_{rj} + g_{rj}\overset{0}{|}{}^l X_{lk}) + \Omega^{ir}_{mj}X^m{}_{rk}, \\ \tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k + X_{lk}\overset{0}{C}{}_i{}^{jl} - \frac{1}{2}\tilde{g}_{ir}(\tilde{g}^{rj}{}_{\overset{0}{|}k} - 2\tilde{\omega}_k\tilde{g}^{rj} + \tilde{g}^{rj}\overset{0}{|}{}^l X_{lk}) + \tilde{\Omega}^{mj}_{ir}\tilde{X}_m{}^r{}_k, \\ \tilde{C}^i{}_j{}^k = \overset{0}{\tilde{C}}{}^i{}_j{}^k + \frac{1}{2}g^{ir}(g_{rj}\overset{0}{|}{}^k - 2\dot{\omega}^k g_{rj}) + \Omega^{ir}_{mj}\tilde{Y}^m{}_r{}^k, \\ C_i{}^{jk} = \overset{0}{C}{}_i{}^{jk} - \frac{1}{2}\tilde{g}_{ir}(\tilde{g}^{rj}\overset{0}{|}{}^k - 2\tilde{\omega}_k\tilde{g}^{rj}) + \tilde{\Omega}^{mj}_{ir}Y_m{}^{rk}, \\ X_{ij\overset{0}{|}k} = 0, \end{cases} \tag{2.4}$$

where: $\omega = \omega_i dx^i + \dot{\omega}^i\delta p_i$, $\tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}^i\delta p_i$ are two arbitrary 1-forms in T^*M, $\overset{0}{|}$ and $\overset{0}{|}$

denote h- and v-covariant derivatives with respect to $\overset{0}{D}{}^*$, *and* X_{ij}, $X^i{}_{jk}$, $\tilde{X}_i{}^j{}_k$, $\tilde{Y}{}^i{}_j{}^k$, $Y_i{}^{jk}$ *are arbitrary tensor fields on* T^*M.

Particular cases:
If $X_{ij} = X^i{}_{jk} = \tilde{X}_i{}^j{}_k = \tilde{Y}{}^i{}_j{}^k = Y_i{}^{jk} = 0$, in Theorem 2.3 we have:

Theorem 2.4. *Let* $\overset{0}{D}{}^*$ *be a given d-linear connection on* T^*M. *Then the following d-linear connection* K^*, *with local coefficients* $K^*\Gamma(N,\omega,\tilde{\omega}) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$ *given by* (2.5) *is conformal metrical with respect to* $\hat{G}$, *corresponding to the 1-forms* ω *and* $\tilde{\omega}$:

$$\begin{cases} H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} + \frac{1}{2}g^{ir}(g_{rj\overset{0}{|}k} - 2\omega_k g_{rj}), \\ \tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k - \frac{1}{2}\tilde{g}_{ir}(\tilde{g}^{rj}{}_{\overset{0}{|}k} - 2\tilde{\omega}_k\tilde{g}^{rj}), \\ \tilde{C}^i{}_j{}^k = \overset{0}{\tilde{C}}{}^i{}_j{}^k + \frac{1}{2}g^{ir}(g_{rj}\overset{0}{|}{}^k - 2\dot{\omega}^k g_{rj}), \\ C_i{}^{jk} = \overset{0}{C}{}_i{}^{jk} - \frac{1}{2}\tilde{g}_{ir}(\tilde{g}^{rj}\overset{0}{|}{}^k - 2\dot{\tilde{\omega}}^k\tilde{g}^{rj}), \end{cases} \tag{2.5}$$

where $\overset{0}{|}$ *and* $\overset{0}{|}$ *denote the h-and v-covariant derivatives with respect to the given d-linear connection,* $\overset{0}{D}{}^*$, *on* T^*M, *and* $\omega = \omega_i dx^i + \dot{\omega}^i\delta p_i$, $\tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}^i\delta p_i$ *are two given 1-forms in* T^*M.

If we take a metrical d-linear connection as $\overset{0}{D}{}^*$ in Theorem 2.4 then (2.5) becomes:

$$\begin{cases} H^i{}_{jk} = \overset{0}{H}{}^i{}_{jk} - \delta^i_j\omega_k, \\ \tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k + \delta^j_i\tilde{\omega}_k, \\ \tilde{C}^i{}_j{}^k = \overset{0}{\tilde{C}}{}^i{}_j{}^k - \delta^i_j\dot{\omega}^k, \\ C_i{}^{jk} = \overset{0}{C}{}_i{}^{jk} + \delta^j_i\dot{\tilde{\omega}}^k. \end{cases} \tag{2.6}$$

As an example of $\overset{0}{D}{}^*$ we take the canonical metrical d-linear connection of G, $\overset{c}{D}{}^*$, given by:

$$\begin{cases} \overset{c}{H}{}^i{}_{jk} = \frac{1}{2}g^{ir}\left(\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{rk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r}\right), \\ \overset{c}{\tilde{H}}{}_i{}^j{}_k = \frac{\partial N_{ik}}{\partial p_j} - \frac{1}{2}\tilde{g}_{ir}\left(\frac{\delta\tilde{g}^{jr}}{\delta x^k} + \frac{\partial N_{lk}}{\partial p_j}\tilde{g}^{lr} + \frac{\partial N_{lk}}{\partial p_r}\tilde{g}^{lj}\right), \\ \overset{c}{\tilde{C}}{}^i{}_j{}^k = \frac{1}{2}g^{ir}\frac{\partial g_{jr}}{\partial p_k}, \\ \overset{c}{C}{}_i{}^{jk} = -\frac{1}{2}\tilde{g}_{ir}\left(\frac{\partial\tilde{g}^{jr}}{\partial p_k} + \frac{\partial\tilde{g}^{rk}}{\partial p_j} - \frac{\partial\tilde{g}^{jk}}{\partial p_r}\right). \end{cases} \tag{2.7}$$

Theorem 2.5. *The following d-linear connection W^*, with local coefficients $W^*\Gamma(\overset{w}{N},\omega,\tilde{\omega}) = (\overset{w}{H}{}^i_{jk}, \overset{w}{\tilde{H}}{}^i_{jk}, \overset{w}{\tilde{C}}{}^{i\,k}_{\,j}, \overset{w}{C}{}_i^{\,jk})$ is conformal metrical with respect to $\hat{G}$:*

$$\begin{cases} \overset{w}{H}{}^i_{jk} = \dfrac{1}{2}g^{ir}\left(\dfrac{\delta g_{jr}}{\delta x^k} + \dfrac{\delta g_{rk}}{\delta x^j} - \dfrac{\delta g_{jk}}{\delta x^r}\right) - \delta^i_j\omega_k - \Omega^{ri}_{jk}\omega_r, \\ \overset{w}{\tilde{H}}{}_i{}^j{}_k = -\dfrac{1}{2}\tilde{g}_{ir}\left(\dfrac{\delta \tilde{g}^{jr}}{\delta x^k} + \dfrac{\delta \tilde{g}^{rk}}{\delta x^j} - \dfrac{\delta \tilde{g}^{jk}}{\delta x^r}\right) + \delta^i_j\tilde{\omega}_k, \\ \overset{w}{\tilde{C}}{}^{i\,k}_{\,j} = \dfrac{1}{2}g^{ir}\left(\dfrac{\partial g_{jr}}{\partial p_k} + \dfrac{\partial g_{rk}}{\partial p_j} - \dfrac{\partial g_{jk}}{\partial p_r}\right) - \delta^i_j\dot{\omega}^k, \\ \overset{w}{C}{}_i^{\,jk} = -\dfrac{1}{2}\tilde{g}_{ir}\left(\dfrac{\partial \tilde{g}^{jr}}{\partial p_k} + \dfrac{\partial \tilde{g}^{rk}}{\partial p_j} - \dfrac{\partial \tilde{g}^{jk}}{\partial p_r}\right) + \delta^j_i\tilde{\omega}^k - 2\Omega^{jk}_{ri}\tilde{\omega}^r, \end{cases} \tag{2.8}$$

where ω and $\tilde{\omega}$ are two given 1-forms in T^*M.

Finally, if we take a conformal metrical d-linear connection with respect to $\hat{G}$ (e.g. W^*) as $\overset{0}{D}{}^*$, in Theorem 2.3, we have:

Theorem 2.6. *Let $\overset{0}{D}{}^*$ be a fixed conformal metrical d-linear connection on T^*M, with local coefficients: $\overset{0}{D}{}^*\Gamma(\overset{0}{N},\omega,\tilde{\omega}) = (\overset{0}{H}{}^i_{jk}, \overset{0}{\tilde{H}}{}_i{}^j{}_k, \overset{0}{\tilde{C}}{}^{i\,k}_{\,j}, \overset{0}{C}{}_i^{\,jk})$. The set of all conformal metrical d-linear connections on T^*M, with respect to $\hat{G}$, corresponding to the 1-forms ω and $\tilde{\omega}$, with local coefficients: $D^*\Gamma(N,\omega,\tilde{\omega}) = (H^i_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^{i\,k}_{\,j}, C_i^{\,jk})$ is given by:*

$$\begin{cases} N_{ij} = \overset{0}{N}_{ij} - X_{ij}, \\ H^i_{jk} = \overset{0}{H}{}^i_{jk} + \left(\overset{0}{\tilde{C}}{}^{i\,l}_{\,j} + \dot{\omega}^l\delta^i_j\right)X_{lk} + \Omega^{ir}_{mj}X^m_{\ rk}, \\ \tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}{}_i{}^j{}_k + \left(\overset{0}{C}{}_i^{\,jl} + \tilde{\omega}^l\delta^i_j\right)X_{lk} + \tilde{\Omega}^{mj}_{ir}\tilde{X}_m{}^r{}_k, \\ \tilde{C}^{i\,k}_{\,j} = \overset{0}{\tilde{C}}{}^{i\,k}_{\,j} + \Omega^{ir}_{mj}\tilde{Y}^{m\,k}_{\ r}, \\ C_i^{\,jk} = \overset{0}{C}{}_i^{\,jk} + \tilde{\Omega}^{mj}_{ir}Y_m{}^{rk}, \\ X_{ij\overset{0}{|}k} = 0, \end{cases} \tag{2.9}$$

where $\omega = \omega_i dx^i + \dot{\omega}^i\delta p_i$, $\tilde{\omega} = \tilde{\omega}_i dx^i + \tilde{\omega}^i\delta p_i$ are two arbitrary 1-forms in T^*M, $\overset{0}{|}$ and $\overset{0}{|}$ denote the h- and v-covariant derivatives with respect to $\overset{0}{D}{}^*$, and X_{ij}, $X^i_{\ jk}$, $\tilde{X}_i{}^j{}_k$, $\tilde{Y}^{i\,k}_{\ j}$, $Y_i^{\,jk}$ are arbitrary tensor fields on T^*M.

If we take $X_{ij} = 0$, in Theorem 2.6 we obtain:

Theorem 2.7. *Let $\overset{0}{D}{}^*$ be a fixed conformal metrical d-linear connection on T^*M, with local coefficients: $\overset{0}{D}{}^*\Gamma(\overset{0}{N},\omega,\tilde{\omega}) = (\overset{0}{H}{}^i_{jk}, \overset{0}{\tilde{H}}{}_i{}^j{}_k, \overset{0}{\tilde{C}}{}^{i\,k}_{\,j}, \overset{0}{C}{}_i^{\,jk})$. The set of all conformal met-*

rical d-linear connections on T^*M, with respect to $\hat{G}$, which preserve the nonlinear connection N, corresponding to the 1-forms ω and $\tilde{\omega}$,with local coefficients: $D^*\Gamma(N,\omega,\tilde{\omega}) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$ is given by:

$$\begin{cases} H^i{}_{jk} = \overset{0}{H^i}{}_{jk} + \Omega^{ir}_{mj} X^m{}_{rk}, \\ \tilde{H}_i{}^j{}_k = \overset{0}{\tilde{H}}_i{}^j{}_k + \tilde{\Omega}^{mj}_{ir} \tilde{X}_m{}^r{}_k, \\ \tilde{C}^i{}_j{}^k = \overset{0}{\tilde{C}^i}{}_j{}^k + \Omega^{ir}_{mj} \tilde{Y}^m{}_r{}^k, \\ C_i{}^{jk} = \overset{0}{C}_i{}^{jk} + \tilde{\Omega}^{mj}_{ir} Y_m{}^{rk}, \end{cases} \tag{2.10}$$

where $X^i{}_{jk}$, $\tilde{X}_i{}^j{}_k$, $\tilde{Y}^i{}_j{}^k$, $Y_i{}^{jk}$ are arbitrary tensor fields on T^*M.

3 Some special classes of conformal metrical d-linear connections.

We shall try to replace the arbitrary tensor fields $X^i{}_{jk}$, $Y_i{}^{jk}$ in Theorem 2.7 by the torsion tensor fields $T^i{}_{jk}, S_i{}^{jk}$.

We put:

$$\begin{cases} T^*{}^i{}_{jk} = \frac{1}{2} g^{ir}(g_{rh}T^h{}_{jk} - g_{jh}T^h{}_{rk} + g_{kh}T^h{}_{jr}), \\ S^*{}_i{}^{jk} = \frac{1}{2} \tilde{g}_{ir}(\tilde{g}^{rh}S_h{}^{jk} - \tilde{g}^{jh}S_h{}^{rk} + \tilde{g}^{kh}S_h{}^{jr}). \end{cases} \tag{3.1}$$

Theorem 3.1. *Let $T^i{}_{jk}$ and $S_i{}^{jk}$ be two given alternate tensor fields and let ω, $\tilde{\omega}$ be two given 1-forms in T^*M. Then there exists a unique conformal metrical d-linear connection with respect to $\hat{G}$, corresponding to the 1-forms ω and $\tilde{\omega}$, with local coefficients: $D^*\Gamma(\overset{w}{N},\omega,\tilde{\omega}) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$, having $T^i{}_{jk}$ and $S_i{}^{jk}$ as the torsion tensor fields. It is given by:*

$$\begin{cases} H^i{}_{jk} = \overset{w}{H^i}{}_{jk} + T^*{}^i{}_{jk}, \\ \tilde{H}_i{}^j{}_k = \overset{w}{\tilde{H}}_i{}^j{}_k, \\ \tilde{C}^i{}_j{}^k = \overset{w}{\tilde{C}^i}{}_j{}^k, \\ C_i{}^{jk} = \overset{w}{C}_i{}^{jk} + S^*{}_i{}^{jk}, \end{cases} \tag{3.2}$$

where $W^*\Gamma(\overset{w}{N},\omega,\tilde{\omega}) = \left(\overset{w}{H^i}{}_{jk}, \overset{w}{\tilde{H}}_i{}^j{}_k, \overset{w}{\tilde{C}^i}{}_j{}^k, \overset{w}{C}_i{}^{jk}\right)$ are the local coefficients of conformal metrical d-linear connection given in (2.8).

Definition 3.1. *A conformal metrical d-linear connection on T^*M, with local coefficients $D^*\Gamma(N,\omega,\tilde{\omega}) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$ is called semi-symmetric conformal metrical d-linear* *connection if the torsion tensor fields* $T^i_{\ jk}$ *and* $S_i^{\ jk}$ *have the form:*

$$\begin{cases} T^i_{\ jk} = \dfrac{1}{n-1}(T_j\delta^i_k - T_k\delta^i_j), \\ S_i^{\ jk} = -\dfrac{1}{n-1}(S^{\ j}\delta^k_i - S^{\ k}\delta^j_i), \end{cases} \tag{3.3}$$

where $T_j = T^i_{\ ji}$ *and* $S^{\ j} = S_i^{\ ji}$.

The conformal metrical d-linear connection W^*, with local coefficients $W^*\Gamma(\overset{w}{N},\omega,\tilde{\omega}) = \left(\overset{w}{H}{}^i_{\ jk}, \overset{w}{\tilde{H}}{}_i{}^j{}_k, \overset{w}{\tilde{C}}{}^i{}_j{}^k, \overset{w}{C}{}_i^{\ jk}\right)$ given in (2.8) is considered as the semi-symmetric conformal metrical d-linear connection, with the vanishing h- and v-torsion vector fields.

Putting:

$$\sigma_j = \frac{1}{n-1}T_j, \quad \tilde{\sigma}^j = \frac{1}{n-1}S^j, \tag{3.4}$$

then (3.1) become:

$$T^{*\,i}_{\ \ jk} = 2\Omega^{ri}_{jk}\sigma_r, \quad S^{*\ jk}_{\ i} = 2\tilde{\Omega}^{jk}_{ri}\tilde{\sigma}^r. \tag{3.5}$$

Using the Theorem 3.1 and the relations (3.5) we have:

Theorem 3.2. *The set of all semi-symmetric conformal metrical d-linear connections with respect to* $\hat{G}$ *, corresponding to the 1-forms* ω *and* $\tilde{\omega}$ *in* T^*M*, with local coefficients:* $D^*\Gamma(\overset{w}{N},\omega,\tilde{\omega},\sigma,\tilde{\sigma}) = (H^i_{\ jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i^{\ jk})$*, is given by:*

$$\begin{cases} H^i_{\ jk} = \overset{w}{H}{}^i_{\ jk} + 2\Omega^{ri}_{jk}\sigma_r, \\ \tilde{H}_i{}^j{}_k = \overset{w}{\tilde{H}}{}_i{}^j{}_k, \\ \tilde{C}^i{}_j{}^k = \overset{w}{\tilde{C}}{}^i{}_j{}^k, \\ C_i^{\ jk} = \overset{w}{C}{}_i^{\ jk} + 2\tilde{\Omega}^{jk}_{ri}\tilde{\sigma}^r, \end{cases} \tag{3.6}$$

where $W^*\Gamma(\overset{w}{N},\omega,\tilde{\omega}) = \left(\overset{w}{H}{}^i_{\ jk}, \overset{w}{\tilde{H}}{}_i{}^j{}_k, \overset{w}{\tilde{C}}{}^i{}_j{}^k, \overset{w}{C}{}_i^{\ jk}\right)$ *are the local coefficients of an arbitrary semi-symmetric conformal metrical d-linear connection,* W^**, given in (2.8) and* $\sigma = \sigma_i dx^i + \dot{\sigma}^i\delta p_i$, $\tilde{\sigma} = \tilde{\sigma}_i dx^i + \dot{\tilde{\sigma}}^i\delta p_i$ *are two arbitrary 1-forms in* T^*M.

4 The group of transformations of conformal metrical d-linear connections.

We study the transformations $D^*\Gamma(N,\omega,\tilde{\omega}) \to \bar{D}^*\Gamma(\bar{N},\omega',\tilde{\omega}')$ of the conformal metrical d-linear connections with respect to $\hat{G}$.

If we replace $\overset{0}{D}{}^*(\overset{0}{N})$ and $D^*\Gamma(N,\omega,\tilde{\omega})$ in Theorem 2.3 by $D^*\Gamma(N,\omega,\omega')$ and $\bar{D}^*\Gamma(\bar{N},\omega',\tilde{\omega}')$, respectively, two conformal metrical d-linear connections, we obtain:

Theorem 4.1. *Two conformal metrical d-linear connections with respect to $\hat{G}$, D^* and $\bar{D}^*$, with local coefficients $D^*\Gamma(N,\omega,\tilde{\omega}) = (H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk})$ and $\bar{D}^*\Gamma(\bar{N},\omega',\tilde{\omega}') = (\bar{H}^i{}_{jk}, \overline{\tilde{H}}_i{}^j{}_k, \overline{\tilde{C}}^i{}_j{}^k, \bar{C}_i{}^{jk})$ respectively, are related as follows:*

$$\begin{cases} \bar{N}_{ij} = N_{ij} - X_{ij}, \\ \bar{H}^i{}_{jk} = H^i{}_{jk} + \tilde{C}^i{}_j{}^l X_{lk} - \delta^i_j p'_k + \delta^i_j \dot{\omega}^l X_{lk} + \Omega^{ir}_{sj} X^s{}_{rk}, \\ \overline{\tilde{H}}_i{}^j{}_k = \tilde{H}_i{}^j{}_k + C_i{}^{jl} X_{lk} + \delta^j_i \tilde{p}'_k - \delta^j_i \dot{\tilde{\omega}}^l X_{lk} + \tilde{\Omega}^{sj}_{ir} \tilde{X}_s{}^r{}_k, \\ \overline{\tilde{C}}^i{}_j{}^k = \tilde{C}^i{}_j{}^k - \delta^i_j \dot{p}'^k + \Omega^{ir}_{sj} \tilde{Y}^s{}_r{}^k, \\ \bar{C}_i{}^{jk} = C_i{}^{jk} + \delta^j_i \dot{\tilde{p}}'^k + \tilde{\Omega}^{sj}_{ir} Y_s{}^{rk}, \\ X_{ij|k} = 0, \end{cases} \tag{4.1}$$

where $p' = \omega' - \omega$, $\tilde{p}' = \tilde{\omega}' - \tilde{\omega}$, $\omega = \omega_i dx^i + \dot{\omega}^i \delta p_i$, $\tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}^i \delta p_i$, $\omega' = \omega'_i dx^i + \dot{\omega}'^i \delta p_i$ and $\tilde{\omega}' = \tilde{\omega}'_i dx^i + \dot{\tilde{\omega}}'^i \delta p_i$, are given 1-forms in T^*M.

Conversely, given the tensor fields X_{ij}, $X^i{}_{jk}$, $\tilde{X}_i{}^j{}_k$, $\tilde{Y}^i{}_j{}^k$, $Y_i{}^{jk}$ and two given 1-forms p' and $\tilde{p}'$ respectively ($p' = p'_i dx^i + \dot{p}'^i \delta p_i$, $\tilde{p}' = \tilde{p}'_i dx^i + \dot{\tilde{p}}'^i \delta p_i$) the above (4.1) is thought to be a transformation of a conformal metrical d-linear connection $D^*\Gamma(N,\omega,\tilde{\omega})$ to a conformal metrical d-linear connection $\bar{D}^*\Gamma(\bar{N},\omega',\tilde{\omega}') = \bar{D}^*\Gamma(\bar{N},\omega+p',\omega'+\tilde{p}')$. We shall denote this transformation by $t(X_{ij}, X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}^i{}_j{}^k, Y_i{}^{jk}, p', \tilde{p}')$.

Thus we have:

Theorem 4.2. *The set C of all transformations $t(X_{ij}, X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}^i{}_j{}^k, Y_i{}^{jk}, p', \tilde{p}')$ given by (4.1) is a transformation group of the set of all conformal metrical d-linear connections with respect to $\hat{G}$, on T^*M, together with the mapping product:*

$$t(X'_{ij}, X'^i{}_{jk}, \tilde{X}'_i{}^j{}_k, \tilde{Y}'^i{}_j{}^k, Y'_i{}^{jk}, p'', \tilde{p}'') \circ t(X_{ij}, X^i{}_{jk}, \tilde{X}_i{}^j{}_k, \tilde{Y}^i{}_j{}^k, Y_i{}^{jk}, p', \tilde{p}') =$$
$$= (X_{ij} + X'_{ij}, X^i{}_{jk} + X'^i{}_{jk}, \tilde{X}_i{}^j{}_k + \tilde{X}'_i{}^j{}_k, \tilde{Y}^i{}_j{}^k + \tilde{Y}'^i{}_j{}^k, Y_i{}^{jk} + Y'_i{}^{jk}, p' + p'', \tilde{p}' + \tilde{p}'').$$

We inquire about the subgroup of transformations of the semi-symmetric conformal metrical d-linear connections.

Let $\overset{w}{N}$ be a given nonlinear connection. Then any semi-symmetric conformal metrical d-linear connection with local coefficients $\bar{D}^*\Gamma(\overset{w}{N},\omega',\tilde{\omega}',\sigma',\tilde{\sigma}') = \left(\bar{H}^i{}_{jk}, \overline{\tilde{H}}_i{}^j{}_k, \overline{\tilde{C}}^i{}_j{}^k, \bar{C}_i{}^{jk}\right)$ with respect to $\hat{G}$ is given by (3.2) with (3.5). Paying attention to (2.8) we have:

Theorem 4.3. *Two semi-symmetric conformal metrical d-linear connections with respect to $\hat{G}$, with local coefficients $D^*\Gamma(\overset{w}{N},\omega,\tilde{\omega},\sigma,\tilde{\sigma}) = \left(H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk}\right)$ and $\bar{D}^*\Gamma(\overset{w}{N},\omega',\tilde{\omega}',\sigma',\tilde{\sigma}') = \left(\bar{H}^i{}_{jk}, \overline{\tilde{H}}_i{}^j{}_k, \overline{\tilde{C}}^i{}_j{}^k, \bar{C}_i{}^{jk}\right)$ are related as follows:*

$$\begin{cases} \bar{H}^i{}_{jk} = H^i{}_{jk} - \delta^i_j p'_k + 2\Omega^{ri}_{jk} q_r, \\ \overline{\tilde{H}}_i{}^j{}_k = \tilde{H}_i{}^j{}_k + \delta^j_i \tilde{p}'_k, \\ \overline{\tilde{C}}^i{}_j{}^k = \tilde{C}^i{}_j{}^k - \delta^i_j \dot{p}'^k, \\ \bar{C}_i{}^{jk} = C_i{}^{jk} + \delta^j_i \dot{\tilde{p}}'^k + 2\tilde{\Omega}^{jk}_{ri} \dot{\tilde{q}}^r, \end{cases} \tag{4.2}$$

where $p' = \omega' - \omega$, $\tilde{p}' = \tilde{\omega}' - \tilde{\omega}$, $q = \sigma' - \sigma - p'$, $\tilde{q} = \tilde{\sigma}' - \tilde{\sigma} - \tilde{p}'$, $p' = p'_i dx^i + \dot{p}'^i \delta p_i$, $\tilde{p}' = \tilde{p}'_i dx^i + \dot{\tilde{p}}'^i \delta p_i$, $q = q_i dx^i + \dot{q}^i \delta p_i$ *and* $\tilde{q} = \tilde{q}_i dx^i + \dot{\tilde{q}}^i \delta p_i$.

Conversely, given 1-forms p', $\tilde{p}'$, q, $\tilde{q}$ in T^*M, the above (4.2) is thought to be a transformation of a semi-symmetric conformal metrical d-linear connection D^*, with local coefficients $D^*\Gamma(\overset{w}{N}, \omega, \tilde{\omega}, \sigma, \tilde{\sigma}) = \left(H^i{}_{jk}, \tilde{H}_i{}^j{}_k, \tilde{C}^i{}_j{}^k, C_i{}^{jk}\right)$, to a semi-symmetric conformal metrical d-linear connection $\bar{D}^*$, with local coefficients $\bar{D}^*\Gamma(\overset{w}{N}, \omega + p', \tilde{\omega} + \tilde{p}', \sigma + p' + q, \tilde{\sigma} + \tilde{p}' + \tilde{q}) = \left(\bar{H}^i{}_{jk}, \overline{\tilde{H}}_i{}^j{}_k, \overline{\tilde{C}}^i{}_j{}^k, \bar{C}_i{}^{jk}\right)$. We shall denote this transformation by: $t(p', \tilde{p}', q, \tilde{q})$.

Thus we have:

Theorem 4.4. *The set* C^s_N *of all transformations* $t(p', \tilde{p}', q, \tilde{q})$ *given by* (4.2) *is a transformations group of the set of all semi-symmetric conformal metrical d-linear connections with respect to* $\hat{G}$, *together with the mapping product:*

$$t(p', \tilde{p}', q, \tilde{q}) \circ t(p'', \tilde{p}'', q', \tilde{q}') = t(p' + p'', \tilde{p}' + \tilde{p}'', q + q', \tilde{q} + \tilde{q}').$$

This group, C^s_N, is an abelian subgroup of $\mathcal{C}$ and acts on the set of all semi-symmetric conformal metrical d-linear connections, having the same nonlinear connection N, transitively.

The transformation $t(p', \tilde{p}', q, \tilde{q}) : D^*\Gamma(N, \omega, \tilde{\omega}, \sigma, \tilde{\sigma}) \to \bar{D}^*\Gamma(N, \omega + p', \tilde{\omega} + \tilde{p}', \sigma + p' + q, \tilde{\sigma} + \tilde{p}' + \tilde{q})$ given by (4.2) is expressed by the product of the following two transformations:

$$\begin{cases} \bar{H}^i{}_{jk} = H^i{}_{jk} - \delta^i_j p'_k, \\ \overline{\tilde{H}}_i{}^j{}_k = \tilde{H}_i{}^j{}_k + \delta^j_i \tilde{p}'_k, \\ \overline{\tilde{C}}^i{}_j{}^k = \tilde{C}^i{}_j{}^k - \delta^i_j \dot{p}'^k, \\ \bar{C}_i{}^{jk} = C_i{}^{jk} + \delta^j_i \dot{\tilde{p}}'^k, \end{cases} \tag{4.3}$$

$$\begin{cases} \bar{H}^i{}_{jk} = H^i{}_{jk} + 2\Omega^{ri}_{jk} q_r, \\ \overline{\tilde{H}}_i{}^j{}_k = \tilde{H}_i{}^j{}_k, \\ \overline{\tilde{C}}^i{}_j{}^k = \tilde{C}^i{}_j{}^k, \\ \bar{C}_i{}^{jk} = C_i{}^{jk} + 2\tilde{\Omega}^{jk}_{ri} \dot{\tilde{q}}^r. \end{cases} \tag{4.4}$$

Definition 4.1. *The transformation* $t : D^*\Gamma(N) \to \bar{D}^*\Gamma(N)$, *of d-linear connections on* T^*M, *defined by* (4.3) *is called co-parallel transformation on* T^*M, *where* p' *and* $\tilde{p}'$ *are two given 1-forms in* T^*M.

Theorem 4.5. *The set C_N^p of all co-parallel transformations, t, given by (4.3) is an abelian group together with the mapping product.*

Definition 4.2. *The transformation $t: D^*\Gamma(N) \to \bar{D}^*\Gamma(N)$, of d-linear connections, given by (4.4) is called Miron transformation by M.Hashiguchi [2], for Finsler spaces.*

Theorem 4.6. *The set, C_N^m, of all Miron transformations, t, given by (4.4) is a transformations group, together with the mapping product.*

Thus we have:

Theorem 4.7. *The group C_N^s , of all transformations $t(p', \bar{p}', q, \bar{q})$ given by (4.2) is the direct product of the group C_N^p , of all co-parallel transformations and the group C_N^m, of all Miron transformations.*

It is noted that the invariants of the group C_N^s, will be the invariants of each of these subgroups and reciprocally.

It is directly shown that by a co-parallel transformation (4.3) the curvature tensor fields $R^i{}_{jkh}$ and $S_i{}^{jkh}$ are transformed as follows:

$$\begin{cases} \bar{R}^i{}_{jkh} = R^i{}_{jkh} - \delta^i_j p'_{kl}, \\ \bar{S}_i{}^{jkh} = S_i{}^{jkh} + \delta^j_i \bar{p}'^{kl}, \end{cases} \tag{4.5}$$

where $p'_{kl}, \bar{p}'^{kl}$ are the components of dp' and $d\bar{p}'$, expressed with respect to D^*

Eliminating $p'_{kl}, \bar{p}'^{kl}$ from (4.5) we have:

$$\bar{R}^{*i}{}_{jkh} = R^{*i}{}_{jkh}, \ \bar{S}^*{}_i{}^{jkh} = S^*{}_i{}^{jkh}, \tag{4.6}$$

where:

$$\begin{cases} R^{*i}{}_{jkh} = R^i{}_{jkh} - \frac{1}{n}\delta^i_j R^s{}_{skh}, \\ S^*{}_i{}^{jkh} = S_i{}^{jkh} - \frac{1}{n}\delta^j_i S_s{}^{skh}. \end{cases} \tag{4.7}$$

Thus we have:

Theorem 4.8. *The tensor fields $R^{*i}{}_{jkh}$ and $S^*{}_i{}^{jkh}$ given by (4.7) are invariants of the group C_N^p.*

Also, we can obtain:

Theorem 4.9. *The following tensor field $C^*{}_i{}^{jk}$, given by (4.8) is an invariant of the group C_N^p:*

$$C^*{}_i{}^{jk} = C_i{}^{jk} - \frac{1}{n}\delta^j_i C_s{}^{sk}. \tag{4.8}$$

In our previous paper [10], starting from the tensor fields:

$$\begin{cases} K^i{}_{jkh} = R^i{}_{jkh} - \bar{C}^i{}_j{}^r \bar{T}_{rkh}, \\ S_l{}^{ijr} = \dfrac{\partial C_l{}^{ij}}{\partial p_r} - \dfrac{\partial C_l{}^{ir}}{\partial p_j} + C_l{}^{kj} C_k{}^{ir} - C_l{}^{kr} C_k{}^{ij}, \end{cases} \tag{4.9}$$

we obtained the following important invariants of the group of semi-symmetric metrical d-linear connections, which preserve the nonlinear connection N, $\overset{ms}{T}_N$, for $n > 2$:

$$\begin{cases} L^i{}_{jkh} = \mathcal{K}^i{}_{jkh} + \dfrac{2}{n-2}\mathcal{A}_{kh}\left\{\Omega^{ri}_{jk}\left(\mathcal{K}_{rh} - \dfrac{\mathcal{K}g_{rh}}{2(n-1)}\right)\right\}, \\ M_i{}^{jkh} = S_i{}^{jkh} + \dfrac{2}{n-2}\mathcal{A}_{kh}\left\{\tilde{\Omega}^{jk}_{ri}\left(S^{rh} - \dfrac{S\tilde{g}^{rh}}{2(n-1)}\right)\right\}, \end{cases} \tag{4.10}$$

where:

$$\mathcal{K}_{jk} = \mathcal{K}^i{}_{jki}, \quad S^{ik} = S_j{}^{ikj}, \quad \mathcal{K} = g^{jk}\mathcal{K}_{jk}, \quad S = \tilde{g}_{ik}S^{ik}. \tag{4.11}$$

If we replace these $\mathcal{K}^i{}_{jkh}$ and $S_l{}^{ijr}$ by the tensor fields $\mathcal{K}^{*i}{}_{jkh}$ and $S^*{}_l{}^{ijr}$ respectively, defined by:

$$\begin{cases} \mathcal{K}^{*i}{}_{jkh} = \mathcal{K}^i{}_{jkh} - \dfrac{1}{n}\delta^i_j\mathcal{K}^m{}_{mkh}, \\ S^*{}_i{}^{jkh} = S_i{}^{jkh} - \dfrac{1}{n}\delta^j_i S_m{}^{mkh}, \end{cases} \tag{4.12}$$

we can obtain the invariants of the group of transformations of semi-symmetric conformal metrical d-linear connections on T^*M, which preserve the nonlinear connection N, C^s_N:

Theorem 4.10. *For $n > 2$ the following tensor fields: $\mathcal{H}^{*i}{}_{jkh}$ and $M^*{}_i{}^{jkh}$ are invariants of the group C^s_N, of transformations, of semi-symmetric conformal metrical d-linear connections on T^*M, which preserve the nonlinear connection N:*

$$\begin{cases} \mathcal{H}^{*i}{}_{jkh} = \mathcal{K}^{*i}{}_{jkh} + \dfrac{2}{n-2}\mathcal{A}_{kh}\left\{\Omega^{ir}_{kj}\left(\mathcal{K}^*{}_{rh} - \dfrac{\mathcal{K}^* g_{rh}}{2(n-1)}\right)\right\}, \\ M^*{}_i{}^{jkh} = S^*{}_i{}^{jkh} + \dfrac{2}{n-2}\mathcal{A}_{kh}\left\{\tilde{\Omega}^{kj}_{ir}\left(S^{*rh} - \dfrac{S^*\tilde{g}^{rh}}{2(n-1)}\right)\right\}, \end{cases} \tag{4.13}$$

where:

$$\mathcal{K}^*{}_{jk} = \mathcal{K}^{*i}{}_{jki}, \quad S^{*ik} = S^*{}_j{}^{ikj}, \quad \mathcal{K}^* = g^{jk}\mathcal{K}^*{}_{jk}, \quad S^* = \tilde{g}_{ik}S^{*ik}. \tag{4.14}$$

Finally we give invariant of the group C^s_N:

Theorem 4.11. *The following tensor field is an invariant of the group C^s_N:*

$$C^*{}_i{}^{jk} - \frac{2}{n-1}\tilde{\Omega}^{kj}_{ir}C^*{}_m{}^{rm}, \tag{4.15}$$

where $C^*{}_i{}^{jk}$ is given by (4.8).

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