LIBERTAS MATHEMATICA, vol XXIV (2004)

Thoughts on Fractals

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Abstract. The purpose of this paper is to focus the scientific interest on the importance of fractals. A fractal is a set M of $\dim M \neq n \in N$. We use the Kolmogoroff definition of the fractalic dimension and give two classical examples of fractals: 1) the famous Cantor's discontinuous, and 2) the von Koch curve. A way of application of fractals in physics is given by L.Nottale [3], who gave an approximation of the fractal function, with resolution ε. The author proposed a model of the matter structure [4,6,7]. The "elementary" particles of the system S_1, are made of some more "elementary" particles a.s.o. The systems of these particles are S_2, $S_3, \ldots$. This construction leads us to a topological space (X, τ), in which the "curves" are neither differentiable, nor continuous. Therefore we propose to develop a theory of motion on such "curves" and a corresponding geometry.

Mathematics Subject Classifications 2000: 28A80; 37F45

1. Generalities. Paraphrasing the first sentence of the "Manifest of the Communist Party" ("A ghost haunts in Europe, the ghost of communism"), we can say that now "A ghost haunts in mathematics, the ghost of fractals."

What is a fractal? A set $\mathcal{M}$ is *a fractal*, if

$$\dim \mathcal{M} \neq n \in N. \tag{1}$$

The definition of the non-integer dimension is not unique. It is not our purpose to talk over this problem, we want only to note the importance of fractals.

One of the possibilities to define the fractalic dimension is that given by Kolmogoroff (see [1], [2]). If we consider an n-dimensional cube with unit side, we can cover this with $N(\varepsilon)$ cubes with the side $\varepsilon = 1/k$ $(k \in N)$. Now, we have

$$\varepsilon^n N(\varepsilon) = 1 \text{ , or } n \ln \varepsilon + \ln N(\varepsilon) = 0. \tag{2}$$

Hence

$$n = \ln N(\varepsilon) / \ln(1/\varepsilon) \tag{3}$$

Generalizing (3), there follows the Kolmogoroff definition of the *fractalic dimension*:

If S is a set, and $N(\varepsilon)$ the smallest number of cubes with the side ε which cover S, then

$$\dim S = \lim_{\varepsilon \to \infty} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}, \tag{4}$$

if this limit exists. If $\dim S$ is non-integer, then S is a *fractal*.

Let us give two examples: a) the famous Cantor's discontinuous C, and b) the von Koch curve K.

a) From the interval $[0, 1]$ we eliminate the open set $(1/3, 2/3)$, and endlessly continuing this procedure in the remaining intervals. In the k-th step we can cover the obtained intervals, each having the length $\varepsilon = 1/3^k$, with 2^k one-dimensional cubes. Hence

$$\dim C = \lim_{k \to \infty} \left(\ln 2^k / \ln 3^k\right) = \ln 2 / \ln 3. \tag{5}$$

b) To obtain the von Koch curve K, we start with the broken line $ABCDE$, the coordinates in plane being $A(0,0)$, $B(1/3,0)$, $C(1/2, \sqrt{3}/2)$, $D(2/3,0)$, $E(1,0)$. This is the generating pattern of the von Koch curve. One replace each of the straight line segments in that generating pattern by a rescaled generating pattern of 4 straight line segments of length $1/3^2$ a.s.o. In the k-th step we can cover this curve with 4^k intervals of length $1/3^k$ and therefore

$$\dim \mathcal{K} = \lim_{k \to \infty} \left(\ln 4^k / \ln 3^k\right) = \ln 4 / \ln 3. \tag{6}$$

2. Fractals in Physics. L. Nottale [3] had supposed that the motion of various physical objects takes place on continuous but nowhere differentiable curves, hence on fractals. In order to save the differential equations of physics, L. Nottale considers, for a real fractalic function $f(x)$, its approximation obtained from smoothing it or averaging at various resolutions

$$f(x, \varepsilon) = \int_{-\infty}^{\infty} \phi(x, y, \varepsilon)\, f(y)\, dy, \tag{7}$$

where $\phi(x, y, \varepsilon)$ is a smoothing function centered on x. Now we impose $\lim_{\varepsilon \to 0} \phi(x, y, \varepsilon) = \delta_{x-y}$, hence

$$\lim_{\varepsilon \to 0} f(x, \varepsilon) = f(x). \tag{8}$$

$f(x, \varepsilon)$ can be everywhere differentiable and $f(x)$ is the so-called *fractal function*. In this treatment we have a new variable (a new dimension), the *resolution* ε

Let us consider an infinitesimal dilatation

$$\varepsilon \to \varepsilon' = \varepsilon\,(1 + d\rho) = \varepsilon + \varepsilon d\rho \tag{9}$$

to the resolution. Omitting the x dependence of a function $\mathcal{L}$ (for example the length) we have

$$\mathcal{L}(\varepsilon') = \mathcal{L}\,(\varepsilon + \varepsilon d\rho) = \mathcal{L}\,(\varepsilon) + \varepsilon \frac{\partial \alpha\,(\varepsilon)}{\partial \varepsilon} d\rho, \tag{10}$$

or

$$\mathcal{L}(\varepsilon') = (1 + D d\rho)\,\mathcal{L}\,(\varepsilon)\,. \tag{11}$$

Here

$$D = \varepsilon \partial / \partial \varepsilon = \partial / \partial \ln \varepsilon \tag{12}$$

since

$$\partial / \partial \varepsilon = (\partial \ln \varepsilon / \partial \varepsilon)\,(\partial / \partial \ln \varepsilon) = (1/\varepsilon)\,\partial / \partial \ln \varepsilon. \tag{13}$$

During the motion on a fractalic curve, because of the non-differentiability, the right and the left directions are not equivalent and therefore appears the Nottales diffusion coefficient. We underline that a lot of results show that the real space-time of physics is fractalic.

3. A hypothesis and a suggestion [4-7]. A half century ago we proposed a model of the matter structure as follows. Let S_1 be the system of the "elementary" particles and s^i an "elementary" particle, where the upper index "i" counts the particles. We suppose that each particle s_1^i is made up of particles s_2^j, whose system is S_2. Going on with this procedure we arrive at the particles $s_3^k, s_4^l, ..., s_n^t, ...$ and to the corresponding systems $S_3, S_4, ..., S_n, ...$. We assume that each particle s_n^i is made up of a finite number of particles s_{n+1}^i. The particles s_n which are constituent particles of a particle s_{n-1} are called *joint particles*, and the others are called *free particles*. The system C_n of all free particles s_n, which is a subsystem of S_n, is called *field of order* n.

Now, we consider a particle s_n^i, from the geometrical view point, as an element e_n^i. The set e_n^i of all the elements determine the *physical space* S. We can consider e_n to be both an element of S and a subset of S.

The sequence of elements $\left\{e_p^{i(p)}\right\}$ with $p = n, n+1, ...$, where $e_p^{i(p)} \supset e_p^{i(p+1)}$, is called *point* P, and we say that P *belongs* to each element of this sequence. The set of all points form the *geometrical space* $\mathbf{X}$.

Let M_n^i be the set of all points which belong to e_n^i. We call M_n^i a *basic set.* If τ is the set whose elements are the basic sets M_p^j and all the sets obtained as a union of basic sets, then X together with τ define the *topological space* (X, τ). The basic sets, which form a basis in (X, τ), are, at the same time, *open, closed, perfect* and *compact*, but the topological space (X, τ) is *not compact.*

We suppose that each of the basic sets M_n^i can be embedded in a closed ball of E_3. Then some subsets A of (X, τ) can be embedded in subsets B of E_3, A will be dense in B, and "$B \setminus A$" will be also dense in B.

Now, the "curves" in (X, τ) are neither differentiable, nor continuous. Therefore we propose to develop a theory of motion on such "curves" and a corresponding geometry.

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