LIBERTAS MATHEMATICA, vol XXIV (2004)

On Kobayashi Dual Metric

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Abstract. This note is a short version of a quite ample work concerning the $\mathcal{L}-$ dual of a complex Finsler space which proves to be a complex Cartan space.

The holomorphic sectional curvature in a complex Cartan space is done here by $\mathcal{L}$-duality. Corroborating with a result of J.Faran ([7]), we prove that the $\mathcal{L}$-dual of the Kobayashi metric coincides with the Finsler-Hamilton Kobayashi metric ([7, 5]).

Mathematics Subject Classification 2000: 53C60, 53B40, 53C55.
Keywords: complex Finsler spaces, Kobayashi metric.

1 Complex Finsler and Cartan spaces

The notion of complex Finsler spaces is well-known ([1, 2, 3, 4, 6, 8, 19]), remarkable results on it being connected with the geometric theory of holomorphic functions. Some geometric aspects in complex Finsler spaces were studied by us ([15, 17]) based on from the more general theory of complex Lagrange spaces.

The notion of complex Cartan space coincides with that of complex Finsler-Hamiltonian space ([5, 7]). Nevertheless, we shall use the term of complex Cartan space as is done in the real case ([11, 12]). In a previous paper we studied the main geometric objects of a complex Cartan space regarding it as particular complex Hamilton space. In the present paper we analyze the extent to which these geometric objects correspond by $\mathcal{L}$-duality to some objects from a complex Finsler space. Some of the results are directly deduced from the more general study of $\mathcal{L}$-dual Hamilton spaces, recently made by us ([18]). Certainly, in this particular case we obtain some interesting specific results.

In the present section the principal notions with which we shall work in complex Finsler and Cartan spaces are briefly recalled.

The base manifold of a complex Lagrange space is the holomorphic bundle $T'M$ of a complex manifold M, and that of a complex Hamilton space is the dual holomorphic bundle T'^*M. Next, $(z^k, \eta^k)_{k=1,n}$ will be the complex coordinates in a local chart in $u \in T'M$ and $(z^k, \zeta_k)_{k=1,n}$ the local coordinates in $u^* \in T'^*M$.

A complex Lagrange space is a pair (M, L), where $L : T'M \to R$ is a regular Lagrangian, that means it fulfills the pseudoconvexity condition: $\det(g_{i\bar{j}}) \neq 0$, with $g_{i\bar{j}}(z, \eta) = \partial^2 L/\partial\eta^i\partial\bar{\eta}^j$ the metric tensor of the space. Assume that L is a positive function which satisfies the homogeneity condition $L(z,\lambda\eta) =\mid \lambda \mid^2 L(z,\eta), \forall(z,\eta) \in T'M - \{0\}$ and $\lambda \in \mathbb{C}$. Then $g = g_{i\bar{j}}dz^i d\bar{z}^j$ is a positive defined form and $\| X \|_z = F(z,X)$ for $F = \sqrt{L}$, is a Minkowski norm. The pair (M,F) is called a complex Finsler space.

In a similar way, a complex Hamilton space (M,H) is defined by the regular Hamiltonian $H : T'^*M \to R$, with $\det(h^{\bar{j}i}) \neq 0$, where $h^{\bar{j}i} = \partial^2 H/\partial\zeta_i\partial\bar{\zeta}_j$ is the metric tensor of the space. Particularly, if H is a positive function, which satisfies the homogeneity condition $H(z,\lambda\zeta) =\mid \lambda \mid^2 H(z,\zeta)$, $\forall(z,\zeta) \in T'^*M - \{0\}$ and $\lambda \in \mathbb{C}$, then the pair $(M,\mathcal{C})$ with $\mathcal{C} = \sqrt{H}$, is a complex Cartan space. The corresponding Minkowski norm is $\| \omega \|_z = \mathcal{C}(z,\omega)$. Certainly, the homogeneity conditions in a complex Finsler space respectively in a complex Cartan space produce some identities as consequences of the Euler Theorem.

In the following, for simplicity, we shall denote by E either the bundle $T'M$ or $(T'M)^*$, according to the case.

A complex nonlinear connection on E, shortly *c.n.c.*, is a splitting of $T'E = HE \oplus VE$, where $VE = \ker(d\pi)$ is the vertical subbundle, with $\pi : E \to M$ the canonical projection. HE is called the horizontal subbundle. The corresponding vertical distributions are spanned by $\left\{\frac{\partial}{\partial\eta^k}\right\}$ or $\left\{\frac{\partial}{\partial\zeta_k}\right\}$, and these are changed with the matrix $\left(\frac{\partial z'^i}{\partial z^j}\right)$, respectively with its inverse. The adapted frames of the *c.n.c.* in the horizontal distributions are of the form:

$$\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^j \frac{\partial}{\partial\eta^j} \quad \text{or} \quad \frac{\delta^*}{\delta z^i} = \frac{\partial}{\partial z^i} - N_{ji}\frac{\partial}{\partial\zeta_j} \tag{1.1}$$

where N_i^j and N_{ji} are the coefficients of the *c.n.c.* on $T'M$, respectively on $(T'M)^*$.

A fixed *c.n.c* determines the decomposition $T_{\mathbb{C}}E = HE \oplus VE \oplus \overline{HE} \oplus \overline{VE}$, and by $h, v, \bar{h}, \bar{v}$, or by $\overset{*}{h}, \overset{*}{v}, \overset{*}{\bar{h}}, \overset{*}{\bar{v}}$, we are denoting the corresponding projectors.

In addition we require that the adapted frames of the *c.n.c.* transform simple by the rule:

$$\frac{\delta}{\delta z^i} = \frac{\partial z'^j}{\partial z^i}\frac{\delta}{\delta z'^j} \quad \text{and} \quad \frac{\delta^*}{\delta z^i} = \frac{\partial z'^j}{\partial z^i}\frac{\delta^*}{\delta z'^j} \tag{1.2}$$

which implies that the coefficients N_i^j and N_{ji} obey certain rules of transformation.

By conjugation the local adapted frames on $T''E$ are obtained. The adapted frames on $T_{\mathbb{C}}E$ will be shortly denoted by: $\delta_k = \frac{\delta}{\delta z^k}$, $\dot{\partial}_k = \frac{\partial}{\partial\eta^k}$ or $\delta^*_k = \frac{\delta^*}{\delta z^k}$, $\dot{\partial}^k = \frac{\partial}{\partial\zeta_k}$ and their conjugates by $\delta_{\bar{k}}$, $\dot{\partial}_{\bar{k}}$ or $\delta^*_{\bar{k}}$, $\dot{\partial}^{\bar{k}}$.

2 Complex Legendre Transformation

In the real case it is well-known the Lagrangian-Hamiltonian formalism from the classical mechanics, this being possible via Legendre transformation.

An excellent study of real corresponding spaces was done by R.Miron ([11, 12]).

In the complex case the situation is a bit more subtle and recently it was solved by us. Some ideas have similitude with the real case for which we refer to the recent book [14].

Let us consider (M,L) a complex Lagrange space. The map $\Phi : T'M \to T''^*M$, defined on an open set $\Phi : U \to U^*$ by

$$\Phi(z^k, \eta^k) = (z^k, \bar{\zeta}_k = \frac{\partial L}{\partial \bar{\eta}^k}) \tag{2.1}$$

is a local diffeomorphism, its matrix being nondegenerate, $\det(g_{j\bar{k}}) \neq 0$. Then, by conjugation the 1-form $\bar{\zeta}_k d\bar{z}^k$ of $T''^*_{u^*}M$ is sent to $\zeta_k dz^k$ of $T''^*_{u^*}M$. Thus, a system of complex coordinates on T'^*M is obtained in a chart included in U^*. The conjugate map $\bar{\Phi} : \bar{U} \to \bar{U}^*$ is a local diffeomorphism of $T''M$ into T'^*M.

Then, the function

$$H = \zeta_i \eta^i + \bar{\zeta}_i \bar{\eta}^i - L \tag{2.2}$$

locally defines a regular Hamiltonian on M.

Definition 2.1. *The local diffeomorphism $\Phi \times \bar{\Phi}$ is called the complex Legendre transformation.*

The inverse $\Phi^{-1} : U^* \to U$, $\Phi^{-1}(z^k, , \bar{\zeta}_k) = (z^k, \eta^k = \frac{\partial H}{\partial \zeta_k})$, determines a Lagrangian structure $L = \zeta_i \eta^i + \bar{\zeta}_i \bar{\eta}^i - H$ and we have $\frac{\partial H}{\partial z^k} = -\frac{\partial L}{\partial z^k}$.

Usually, by "*" will be denoted the image by Φ of various geometric objects and by "o" their image by Φ^{-1}. The properties obtained by Φ or by Φ^{-1} are said obtained by $\mathcal{L}-$*duality*, and these are locally valid if in addition it is not required something else.

Proposition 2.1. ([18]) *By $\mathcal{L}-$duality we have:*

$$(g^{\bar{m}j})^* = h^{\bar{m}j} \; ; \; (h_{i\bar{m}})^o = g_{i\bar{m}} \tag{2.3}$$

The tangent map $d\Phi$ sends the frames $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \bar{\eta}^i}\}$ into $\{(\frac{\partial}{\partial z^i})^*, (\frac{\partial}{\partial \bar{z}^i})^*, (\frac{\partial}{\partial \eta^i})^*, (\frac{\partial}{\partial \bar{\eta}^i})^*\}$ and therefore the $\mathcal{L}-$*dual* of the adapted base $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}\}$ will be $\{(\frac{\delta}{\delta z^k})^*\}$. In [18] we prove that there exists a unique way such that $\{(\frac{\delta}{\delta z^k})^*\}$ are adapted frames of a *c.n.c* on (M, H) and hence, locally, it spans a horizontal distribution HT'^*M:

Theorem 2.1. ([18]) *There exists only one pair of c.n.c. on $T'M$ and on T'^*M respectively, which correspond by $\mathcal{L}-$duality:*

$$N^i_j = g^{\bar{k}i} \frac{\partial^2 L}{\partial z^j \partial \bar{\eta}^k} \quad and \quad N_{ji} = -h_{j\bar{k}} \frac{\partial^2 H}{\partial z^i \partial \bar{\zeta}_k} \; . \tag{2.4}$$

These *c.n.c* are important in our study because the brackets of their adapted frames are $[\delta_j, \delta_k] = 0$ and $[\delta^*_j, \delta^*_k] = 0$. In the next, everywhere will be used only this pair of $\mathcal{L}-$dual *c.n.c.*

We shall make an important change of frame $\frac{\partial}{\partial \zeta^i} = h_{i\bar{k}} \frac{\partial}{\partial \bar{\zeta}_k}$. Then we have:

Proposition 2.2. i) $(\frac{\delta}{\delta z^i})^* = \frac{\delta^*}{\delta z^i}$; $(\frac{\partial}{\partial \eta^i})^* = \frac{\partial}{\partial \zeta^i}$; $(\frac{\delta^*}{\delta z^i})^o = \frac{\delta}{\delta z^i}$; $(\frac{\partial}{\partial \zeta_i})^o = \frac{\partial}{\partial \eta_i}$
ii) $(dz^i)^* = d^*z^i$; $(\delta\eta^i)^* = \delta\zeta^i = h^{\bar{j}i}\delta\bar{\zeta}_j$; $(d^*z^i)^o = dz^i$; $(\delta\zeta^i)^o = \delta\eta^i$
iii) *If J and J^* are the complex structures on $T_{\mathbb{C}}M$ and on $T^*_{\mathbb{C}}M$ respectively, then*

$(v)^* = \overset{*}{\bar{v}}$; $(\bar{v})^* = \overset{*}{v}$; $(h)^* = \overset{*}{h}$; $(\bar{h})^* = \overset{*}{\bar{h}}$

$(Jv)^* = -\overset{*}{J}\overset{*}{\bar{v}}$; $(J\bar{v})^* = -\overset{*}{J}\overset{*}{v}$; $(Jh)^* = \overset{*}{J}\overset{*}{h}$; $(J\bar{h})^* = \overset{*}{J}\overset{*}{\bar{h}}$

iv) $(X^*)^o = X$; $(fX + gY)^* = f^*X^* + g^*Y^*$; $[X, Y]^* = [X^*, Y^*]$.

Now let us find the image by $\mathcal{L}$–duality of the complex Finsler function $F = \sqrt{L}$. First of all from (2.2) we get $H(z, \frac{\partial L}{\partial \eta}) = L(z, \eta)$. That means $(H(z,\zeta))^o = L(z,\eta)$ and conversely $(L(z,\eta))^* = H(z,\zeta)$. Then we have:

$$H(z, \lambda\zeta) = H(z, \lambda\frac{\partial L}{\partial \eta}) = \left(L(z, \bar{\lambda}\eta)\right)^* = \mid \bar{\lambda} \mid^2 \left(L(z,\eta)\right)^* = \mid \lambda \mid^2 H(z,\zeta).$$

Clearly, it proves that

Proposition 2.3. *The $\mathcal{L}$–dual of a complex Finsler space (M, F) is locally a complex Cartan space $(M, \mathcal{C})$.*

Let us consider the Sasaki lift **G** and **H** of $g_{i\bar{j}}$ and $h^{\bar{j}i}$ respectively the metric tensors.

Proposition 2.4. *The $\mathcal{L}$–dual of the metric tensor and metric structure of a complex Finsler space are:*

$$\left(g_{i\bar{j}}(z,\eta)\right)^* = h_{i\bar{j}}(z,\zeta) \ ; \ (\mathbf{G}(X,Y))^* = \mathbf{H}(X^*, Y^*).$$

That means that the complex Legendre transformation Φ is locally an isometry. The length on (M, F), $\parallel X_z \parallel_F$ corresponds by $\mathcal{L}$–duality with the length on $(M, \mathcal{C})$, $\parallel Y_z \parallel_{\mathcal{C}} = \parallel Y_z^o \parallel_F$

In the complex Finsler space (M, F) it is well-known a complex linear connection, shortly *c.l.c*, Hermitian and of $(1,0)$–type, called the Chern-Finsler complex connection. This is denoted by $D\Gamma \overset{CF}{N}$ and has the local coefficients:

$$\overset{CF}{N_j^i} = g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l \ ; \ \overset{CF}{L_{jk}^i} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k} = \frac{\partial \overset{CF}{N_k^i}}{\partial \eta^j} \ ; \ \overset{CF}{C_{jk}^i} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^k} \tag{2.5}$$

Let us note that $\overset{CF}{N_j^i}$ coincides with our canonical *c.n.c* from (2.4).

Analogously in a complex Cartan space there exists an unique N^* – *c.l.c.*, Hermitian and of $(1,0)$–type, namely the Chern-Cartan connection $D^*\Gamma \overset{CC}{N}$ of local coefficients :

$$\overset{CC}{N_{ji}} = -h_{j\bar{m}}\frac{\partial h^{\bar{m}l}}{\partial z^i}\,\zeta_l; \ \overset{CC}{H_{jk}^i} = -h_{j\bar{m}}\frac{\delta^* h^{\bar{m}i}}{\delta z^k} = \frac{\partial \overset{CC}{N_{jk}}}{\partial \zeta_i} \ ; \ \overset{CC}{C_j^{ik}} = -h_{j\bar{m}}\frac{\partial h^{\bar{m}i}}{\partial \zeta_k} \tag{2.6}$$

Let us note again that $\overset{CC}{N_{ji}}$ coincides with the canonical *c.n.c* from (2.4).

The study of the $\mathcal{L}$–dual of the complex Chern-Finsler connection is interesting too. We can prove that the $\mathcal{L}$–dual of the complex Chern-Finsler connection is an Hermitian connection but not of $(1,0)$– type and hence it is not the Chern-Cartan connection excepting the case when the Finsler metric comes from an Hermitian one on the base manifold M.

3 The $\mathcal{L}$–dual holomorphic sectional curvature

It is known that in a complex Finsler space (M, F) there are two ways for defining the holomorphic sectional curvature.

The first arrives to the holomorphic sectional curvature on (M, F) from the Gauss curvature on the unit disc $\Delta \subset \mathbb{C}$ via a holomorphic function and is a definition at the $T'M$ level.

The second is closely related to the classical definition of the holomorphic curvature on a complex manifold and was considered by S. Kobayashi ([4]). Let $\eta = \eta^k \frac{\partial}{\partial z^k} \in T'_z M$ be a tangent vector and $\{\frac{\delta}{\delta z^k}\}$ the adapted frame with respect to the Chern-Finsler *c.n.c.* Then the holomorphic sectional curvature in a direction η is:

$$K_F(z,\eta) = \frac{2}{L(z,\eta)} \mathbf{G}\left(R(\chi,\bar{\chi})\chi,\bar{\chi}\right) \tag{3.1}$$

where $\chi = l^h(\eta) = \eta^k \frac{\delta}{\delta z^k}$ is the horizontal lift of η and $\mathbf{G}$ is the Sasaki lift of Finsler metric. This is a definition at the $T'(T'M)$ level. It is proved that the Gaussian sectional holomorphic curvature is bounded upper by $K_F(z,\eta)$ ([1, 3]).

The notion of complex Finsler geodesic is also very important in our theory and we recall it in the next lines.

Let Δ_r be the $r-$complex disc and $\mu_c = \frac{1}{1+\frac{c}{2}|w|^2} dw d\bar{w}$ a Hermitian metric on Δ_r , which particularly contains the Poincaré metric for $c = -2$, the euclidean complex metric for $c = 0$ and the Fubini-Study metric for $c = 2$. The equations of a geodesic σ with respect to μ_c are $\ddot{\sigma} = A_c(w)\dot{\sigma}^2$, where $A_c(w) = \frac{c\bar{w}}{1+\frac{c}{2}|w|^2}$.

A $c-$segment of complex geodesic on (M,F) is defined by Abate-Patrizio ([1]) as being a holomorphic map which sends a geodesic on Δ_r, parameterized by arc length with respect to μ_c, into a geodesic on (M,F), parameterized by arc length with respect to F. We shall use for our main result the next theorem.

Theorem 3.1. ([1, p.131]) *φ is a $c-$segment of complex geodesic on (M,F), $c \in \mathbb{R}$, ($r < \sqrt{2/\mid c \mid}$ for $c < 0$), iff*

$$D_{\varphi'^h} \varphi'^h = -A_c \chi(\varphi') \tag{3.2}$$

$$\mathbf{G}(hT(X^h, \varphi'^h), \overline{\varphi'^h}) = 0; \quad \forall X^h \in HT'M \tag{3.3}$$

A complex Finsler space in which the condition (3.3) is fulfilled is said to be weakly Kähler along φ.

Let be $S'M = \{\eta \in T'M \ / \ F(z,\eta) = 1\}$ be the unit sphere bundle.

Theorem 3.2. ([1, p.141]) *There exists an unique $c-$geodesic complex curve through (z,η) for any $(z,\eta) \in S'M$, iff (M,F) is weakly Kähler and*

$$vT(\varphi'^h, \overline{\varphi'^h}) = c\mathbf{G}(\varphi'^h, \overline{\varphi'^h}) l^v(\varphi') \tag{3.4}$$

where $l^v(\varphi') = \varphi'^k \frac{\partial}{\partial \eta^k}$ is the vertical lift of φ'.

When the geodesic exits it is unique.

Using the Ahlfors' Lemma for an upper semicontinuous function a fundamental result concerning the Kobayashi metric

$$F_M(z,\eta) = \inf \left\{ \frac{1}{r} \ / \ \exists \varphi : \Delta_r \to M, \text{ holom.}, \varphi(0) = z, \ \varphi'(0) = \eta \right\} \tag{3.5}$$

is obtained, and this result is given in a suitable form for us in Theorem 3.1.15, p.146, from [1], (see also [6]):

Theorem 3.3. *Let $F : T'M \to R_+$ be a strongly pseudoconvex smooth and complete metric on a complex manifold M. Assume that one of these equivalent conditions holds:*

i) *The conditions (3.3) and (3.4) for $c = -2$*

ii) $K_F \equiv -4$ *and* $\mathbf{G}(\bar{\partial}\theta(X^h, \chi, \bar{\chi}), \bar{\chi}) = 0$, $\forall X^h \in HT'M$

iii) $K_F \equiv -4$ *and* $\mathbf{G}(\Omega(X^h, \bar{\chi})\chi, \bar{\chi}) = \mathbf{G}(\Omega(\chi, \bar{\chi})X^h, \chi)$, $\forall X^h \in HT'M$

Then F is the Kobayashi metric of M.

(θ and Ω are components of Chern-Finsler curvature and torsion, respectively).

These equivalent conditions include the fact that the indicatrix is convex and this is a condition for the existence of minimal geodesics in the weakly Kähler case.

Now let us transfer the problem of sectional curvatures by $\mathcal{L}$-duality in a complex Cartan space.

We make an additional hypothesis for the complex Legendre transformation, namely that $\Phi : U \to U^*$ contains in any $(z, \eta) \in U$ the ball $S'M$. Clearly, the condition "Φ globally defined" is sufficiently for us.

Let $g_{i\bar{j}}$ be the metric tensor in the complex Finsler space (M, F) and $h^{\bar{j}i}$ be the $\mathcal{L}$-dual metric tensor in the complex Cartan space $(M, \mathcal{C} = (F)^*)$. A curve in a complex Cartan space is obtained by the composition of the tangent map $\varphi' : \Delta \to T'M$, $\varphi'(w) = (z^k = \varphi^k(0), \eta^k = \frac{d\varphi^k}{dw}(0))_{k=1,n}$ with Φ, and then we give the map $\varphi'^* = \Phi \circ \varphi' : \Delta \to T'^*M$, $\varphi'^*(w) = (z^{*k}, \zeta_k(w))_{k=1,n}$, where $z^{*k} = \varphi^{*k}(0) = \Phi(\varphi^k)(0)$ and $\zeta_k(w)) = \left(\frac{\partial L}{\partial \eta^k}\right)^*(0) \doteq g^*_{k\bar{j}}(w)\overline{(\varphi'^j)^*}(0)$, with $g^*_{k\bar{j}}(w) = (g_{k\bar{j}}(w))^*$. From the above construction it results that $h = (g)^*$.

On the other hand, consider the morphism $l^{*h} : T'^*M \to H(T'^*M)$ given by $l^{*h} = d\Phi \circ l^h \circ \Phi$, which in any $z \in U$ send the 1-form $\zeta = \zeta_k dz^k$ into the horizontal vector $l^{*h}(\zeta) = h^{\bar{j}k}\bar{\zeta}_j\delta^*_k = \zeta^k\delta^*_k = \chi^*$. Such a morphism allows us to define the holomorphic sectional curvature on $(M, \mathcal{C} = (F)^*)$ with respect to ζ as being

$$K_{F^*}(z, \zeta) = \frac{2}{H(z, \eta)}\mathbf{H}\left(R^*(\chi^*, \bar{\chi}^*)\chi^*, \bar{\chi}^*\right) \tag{3.6}$$

By $\mathcal{L}$-duality is obtained a right definition of the arc length of a curve on $(M, \mathcal{C} = (F)^*)$ as being $l_{\Gamma^*} = \int_a^b F^*(z^k(t), g^*_{k\bar{j}}(t)\frac{d\bar{z}^j}{dt})dt$ and likewise in the complex Finsler case is defined the notion of $c-$ segment of complex geodesic in the Cartan space $(M, \mathcal{C})$.

Since torsion $(T)^* = T^*$, Theorem 3.1, which characterize the $c-$ segment of complex Finsler geodesic, is translated without difficulty to the $\mathcal{L}$-dual Cartan space. The correspondent of the second condition from Theorem 3.1 is $\mathbf{H}(hT^*(X^{*h}, \varphi'^{*h}), \overline{\varphi'^{*h}}) = 0$; $\forall X^h \in HT'^*M$ and this is just the condition to be weakly Kähler Cartan space. Therefore we have:

Theorem 3.4. *In a weakly Kähler complex Cartan space $\mathcal{L}$-dual of (M, F) there exists an unique complex geodesic through any $(z, \zeta) \in S'^*M$ iff:*

$$\overset{*}{\bar{v}}T^*(\varphi'^{*h}, \overline{\varphi'^{*h}}) = c\mathbf{H}(\varphi'^{*h}, \overline{\varphi'^{*h}})l^{*v}(\varphi'^*) \tag{3.7}$$

where $l^{*v}(\varphi'^*) = \varphi'^{*k}h_{k\bar{j}}\frac{\partial}{\partial\zeta_j}$ is the vertical lift on T'^*M.

Let us consider the $\mathcal{L}-$dual of Kobayashi metric (3.3) defined on a open set in U which assure its differentiability condition:

$$F_M^*(z^*,\zeta)=(F_M(z,\eta))^*=\inf_{\varphi\ holom}\left\{\frac{1}{r}\ /\ \exists\varphi:\Delta_r\to M,\varphi^*(0)=z^*,\varphi'^*(0)=\zeta\right\}$$

where $z^{*k}=\Phi(\varphi^k(0))$ is identified with z^k.

As Φ is a diffeomorphism on U it follows that F_M^* fulfills the conditions of differentiability, positivity and pseudoconvexity ($h^{\bar{l}m}\zeta_m\bar{\zeta}_l>0$).

Now, taking into account all the following reasons : i) $(K_F)^*=K_{F^*}$, $\mathbf{H}=(\mathbf{G})^*$, ii) the horizontal induced curvature is $\Omega^*=(\Omega)^*$, iv) $F(z,\eta)\leq 1$ implies $F^*(z,\zeta)\leq 1$; it results:

Theorem 3.5. *Let (M,F) be a complex Finsler space satisfying the conditions from Theorem 3.3. If $(M,\mathcal{C}=(F)^*)$ is the $\mathcal{L}-$dual complex Cartan space, then F^* coincides with F_M^*.*

Finally, let us recall that in [5], S. Kobayashi studied the following metric on T'^*M :

$$\mathcal{C}_M(z,\zeta)=\sup\parallel f^*(\zeta)\parallel_P \tag{3.8}$$

where $\parallel f^*(\zeta)\parallel_P$ is the length of a cotangent vector measured by Poincaré metric on the unit disc Δ, and the supremum is taken over all holomorphic functions $f:\Delta\to M$, $f(0)=z$. The metric (3.8) is named in [7] the Kobayashi Finsler-Hamiltonian metric.

In [7], J. Faran proves by the hard technique of Cartan moving frame (the equivalence problem) the next result in our terminology:

Theorem 3.6. *Let M be a complex manifold and $\mathcal{C}$ a complete, weakly Kähler, complex Cartan space on M with strongly convex indicatrix and constant holomorphic curvature -4 satisfying the additional condition that $F^\alpha=0$. Then $\mathcal{C}$ is the Kobayashi metric $\mathcal{C}_M$ on T'^*M.*

Note that in Faran's work, $F^\alpha=0$ means that some torsion components vanish and this is equivalent with (3.7) because it is $\mathcal{L}-$ dual of $E^\alpha=0$ from [6].

Consequently the following important result follows:

Theorem 3.7. *Under the conditions of Theorem 3.3 the Kobayashi dual metric $\mathcal{C}_M$ coincides with the image by $\mathcal{L}-$duality of Kobayashi metric F_M, i.e. $F_M^*\equiv\mathcal{C}_M$.*

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