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Singular Hamilton Spaces

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Abstract. One studies the singular metrics determined by non-regular Hamiltonians on cotangent bundle and compatible d-connections with singular metrics.

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1 Singular Hamilton Spaces $S\mathcal{H}^n = (M, \mathcal{H}(x,p))$.

Let (T^*M, τ^*, M) be the cotangent bundle of a C^∞ -differentiable real, n-dimensional manifold M. If (U, φ) is a local chart on M then the coordinates of a point $u = (x,p) \in \tau^{*-1}(U) \in T^*M$ will be denoted by (x^i, p_i), $(i = \overline{1,n})$. A differentiable Hamiltonian on T^*M is a mapping

$$\mathcal{H} : (x,p) \in T^*M \to \mathcal{H}(x,p) \in \mathbb{R},$$

of class C^∞ on $\widetilde{T^*M} = T^*M \setminus \{0\}$ and continuous on null section of the projection $\tau^* : T^*M \to M$.

Definition 1. *A differentiable Hamiltonian $\mathcal{H}$ on T^*M is said to be non-regular if the matrix*

$$g^{ij}(x,p) = \frac{1}{2}\frac{\partial^2 \mathcal{H}(x,y)}{\partial p_i \partial p_j} \tag{1}$$

has the rank $n - k > 0$, where k is a natural number and $0 < k < n$.

Definition 2. *A singular Hamilton space is a pair $S\mathcal{H}^n = (M, \mathcal{H}(x,y))$ which consist of a smooth manifold M and a non-regular Hamiltonian $\mathcal{H}$ for which the quadratic form $\Psi = g^{ij}(x,p)\xi_i\xi_j$, has constant signature on $\widetilde{T^*M}$.*

We consider the vertical singular metric G^V on T^*M

$$G^V\left(\partial^i, \partial^j\right) = g^{ij}(x,p) \tag{2}$$

where $\partial^i = \frac{\partial}{\partial p_i}$ and the distribution V_1 given by

$$V_1 = \{\xi_j \in \mathcal{X}(T^*M) \mid g^{ij}\xi_j = 0\} \tag{3}$$

Because $rank(g^{ij}) = n - k$, locally there exist k linearly independent solution of the equation $g^{ij}\xi_j = 0$ given by

$$\xi^a = \xi^a_i \partial^i, \quad (a, b, c, \ldots = \overline{1,k}) \tag{4}$$

Let V_2 be a complementary distribution to V_1 in the vertical distribution V on T^*M, where V_2 is generated by $n-k$ vector fields ς^α such as

$$\varsigma^\alpha = \varsigma_i^\alpha \partial^i, \quad (\alpha, \beta, \ldots = \overline{k+1, n}) \tag{5}$$

Consequently, we have $rank\begin{pmatrix}\xi_i^a & \varsigma_i^\alpha\end{pmatrix} = n$ and

$$V = V_1 \oplus V_2 \tag{6}$$

Let $A^{-1} = \begin{pmatrix}\eta_a^i \\ \theta_\alpha^i\end{pmatrix}$ be the inverse of the matrix $A = \begin{pmatrix}\xi_i^a & \varsigma_i^\alpha\end{pmatrix}$. We consider the following d–tensor fields

$$l_j^i(x,y) = \xi_j^a \eta_a^i, \quad m_j^i(x,y) = \varsigma_j^\alpha \theta_\alpha^i, \tag{7}$$

We have the relations

$$\begin{cases} l_j^i + m_j^i = \delta_j^i \Leftrightarrow l + m = \delta \\ l_j^i l_k^j = l_k^i \Leftrightarrow l^2 = l, \; m_j^i m_k^j = m_k^i \Leftrightarrow m^2 = m \\ l_j^i m_k^j = 0 \Leftrightarrow lm = 0, \; m_j^i l_k^j = 0 \Leftrightarrow ml = 0 \end{cases} \tag{8}$$

and l_j^i, m_j^i are locally projectors on V_1 and V_2, respectively.

Proposition 1. *The projectors l_j^i, m_j^i satisfy the following relation*

$$g^{ij} l_i^k = 0, \quad g^{ij} m_i^k = g^{jk}. \tag{9}$$

Now, we consider the matrix

$$\overline{g} = \begin{pmatrix} g^{ij} & {}^t\eta_a^i \\ \eta_a^i & 0 \end{pmatrix} \tag{10}$$

which is symmetric. The matrix $\overline{g}$ is not singular.

Theorem 1. *The inverse of the matrix $\overline{g}$ is the matrix*

$$\overline{g}^{-1} = \begin{pmatrix} \widetilde{g}_{ij} & \xi_a^i \\ {}^t\xi_a^i & 0 \end{pmatrix} \tag{11}$$

where the matrix $\widetilde{g}_{ij}$ is uniquely determined by

$$\widetilde{g}_{ij} g^{ir} = m_j^r, \quad l_i^j \widetilde{g}_{jr} = 0. \tag{12}$$

With respect to the singular metric $g^{ij}(x,p)$ we introduce Obata-Oproiu operators $\overset{1}{\Phi}$, $\overset{2}{\Phi}$, $\overset{3}{\Phi}$ ([8]):

$$\begin{cases} \overset{1}{\Phi}{}_{ij}^{pq} = \frac{1}{2}\left(\delta_i^p \delta_j^q + l_i^p l_j^q - \widetilde{g}_{ij} g^{pq}\right), \\ \overset{2}{\Phi}{}_{ij}^{pq} = \frac{1}{2}\left(\delta_i^p \delta_j^q - l_i^p l_j^q + \widetilde{g}_{ij} g^{pq}\right), \\ \overset{3}{\Phi}{}_{ij}^{pq} = \frac{1}{2}\left(l_i^p m_j^q + m_i^p l_j^q\right). \end{cases} \tag{13}$$

Considering

$$\overset{1}{O} = \overset{1}{\Phi} - \overset{3}{\Phi}, \quad \overset{2}{O} = \overset{2}{\Phi} + \overset{3}{\Phi} \tag{14}$$

we get

$$\overset{1}{O} + \overset{2}{O} = I, \quad \overset{a}{O}{}^2 = \overset{a}{O}, \quad \overset{1}{O}\overset{2}{O} = \overset{2}{O}\overset{1}{O} = 0, \quad (a = 1,2), \tag{15}$$

so $\overset{1}{O}$ and $\overset{2}{O}$ are the supplementary projectors on the module of d-tensor fields of type $(1,2)$, and are expressed as

$$\overset{1}{O}{}_{ij}^{pq} = \frac{1}{2}\left(\delta_i^p\delta_j^q - \delta_j^q l_i^p - \delta_i^p l_j^q + 3l_i^p l_j^q - \widetilde{g}_{ij}g^{pq}\right) \tag{16}$$

$$\overset{2}{O}{}_{ij}^{pq} = \frac{1}{2}\left(\delta_i^p\delta_j^q + \delta_j^q l_i^p + \delta_i^p l_j^q - 3l_i^p l_j^q + \widetilde{g}_{ij}g^{pq}\right) \tag{17}$$

2 Compatible connections with singular metric tensor fields

Let $D(N) = (H_{jk}^i, C_k^{ij})$ be a d-linear connection in singular Hamilton space, where non-linear connection N is fixed. We denote with $_{/}$ and with $/$ the h-and v-covariant derivatives with respect to $D(N)$.

Definition 3. *A d-connection $D(N)$ is called compatible with a singular metric $g^{ij}(x,p)$ if*

$$g^{ij}_{/k} = 0, \quad g^{ij}/^k = 0, \quad l^h_{i/k} = 0, \quad l^h_i/^k = 0. \tag{18}$$

Proposition 2. *For the d-connection $D(N)$ which is compatible with a singular metric $g^{ij}(x,p)$ we have*

$$m^i_{j/k} = 0, \quad m^i_j/^k = 0, \quad \widetilde{g}_{ij/k} = 0, \quad \widetilde{g}_{ij}/^k = 0. \tag{19}$$

In order to determine the existence of d-connections compatible with a singular metric g^{ij}, we consider an arbitrary but fixed d-connection $D(\overline{N}) = (\overline{H}_{jk}^i, \overline{C}_k^{ij})$ being fixed, and generalize the *Kawaguchi method* to this metric. We have:

Theorem 2. *The d-connection $D(N) = (H^{\ i}_{jk}, C^{\ ij}_{k})$ given by*

$$\begin{cases} N_{ij} = \overline{N}_{ij} \\ H_{jk}^i = \overline{H}_{jk}^i + V_{jk}^i \\ C_j^{ik} = \overline{C}_j^{ik} + \widetilde{V}_j^{ik} \end{cases} \tag{20}$$

where

$$V_{jk}^i = \frac{1}{2}(g^{ir}\widetilde{g}_{rj\overline{/}k} + 3l_r^i l^r_{j\overline{/}k} - l^i_{j\overline{/}k}), \tag{21}$$

$$\widetilde{V}_i^{jk} = -\frac{1}{2}(\widetilde{g}_{is}g^{js}\overline{/}^k + 3l_i^r l_r^j\overline{/}^k - l_i^j\overline{/}^k). \tag{22}$$

is compatible with singular metric $g^{ij}(x,p)$, where $(\overline{N}_{ij}, \overline{H}_{jk}^i, \overline{C}_j^{ik})$ is an arbitrary fixed d-connection.

Theorem 3. *The following d-connection $\overset{c}{D}(N) = (\overset{c}{H}{}^h_{jk}, \overset{c}{C}{}^{ij}_k)$ given by*

$$\begin{cases} \overset{c}{H}{}^h_{jk} = \partial^h N_{jk} + \frac{1}{2}\left(g^{ih}\widetilde{g}_{ij\overline{/}k} + 3l_t^h l^t_{j\overline{/}k} - l^h_{j\overline{/}k}\right), \\ \overset{c}{C}{}^{jk}_i = -\frac{1}{2}\left(\widetilde{g}_{is}\partial^k g^{js} + 3l_i^r\partial^k l_r^j - \partial^k l_i^j\right), \end{cases} \tag{23}$$

is compatible with singular metric tensor $g^{ij}(x,p)$ and is called the canonical metrical connection of the singular Hamilton space.

Theorem 4. *The set of all d-connection $D(N) = (H^i_{jk}, C^{ij}_j)$ compatible with the singular metric g^{ij} is given by*

$$\begin{cases} H^h_{jk} = \overset{c}{H}{}^h_{jk} + \overset{1}{O}{}^{ih}_{jr} Y^r_{ik}, \\ C^{jk}_i = \overset{c}{C}{}^{jk}_i + \overset{1}{O}{}^{js}_{ri} Z^{rk}_s \end{cases} \tag{24}$$

where $\overset{c}{D}(N) = (\overset{c}{H}{}^i_{jk}, \overset{c}{C}{}^{ij}_k)$ is the canonical metrical d-connection and Y^h_{jk}, Z^{hs}_k are arbitrary d-tensor fields.

Remark 1. If $k = 0$, then $l = 0$, $m = \delta$ and $\widetilde{g} = g^{-1}$ one obtains the case studied by Acad. R. Miron in [6].

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