LIBERTAS MATHEMATICA, vol XXIV (2004)

On Totally Contact Umbilical QR-Submanifolds of a Manifold with a Generalized 3-Sasakian Structure

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Abstract. We obtain a classification of totally contact umbilical QR-submanifolds of a manifold with a generalized Sasakian 3-structure (cf. Lemma 3.1, Theorem 3.1).

AMS Subject Classification: 53C40, 53C42.

Keywords: QR-submanifolds; totally contact submanifolds; extrinsic sphere.

Introduction

The notion of CR-submanifold has been introduced by A.Bejancu [2] for the Kähler manifolds, by A.Bejancu-N.Papaghiuc [4] for the Sasakian manifolds (called contact semi-invariant submanifolds).

In the case of quaternionic Kähler manifolds it has been introduced the concept of CR-submanifolds by Baros-Chen-Urbano [1], by using the decomposition of the tangent bundle of a submanifold and the concept of QR-submanifolds by A.Bejancu [3] by using the decomposition of the normal bundle of a submanifold. Later, the notion of CR-submanifolds has been intensively studied from different points of view, several important results have been obtained, some of them being brought together in [2]. Also some important results have been obtained in [3] on QR-submanifolds of quaternionic Kaehlerian manifolds and in [7] above semi-invariant submanifolds of a manifold with a Sasakian 3-structure.

Recently A.Bejancu-H.R.Faran [5] have introduced the notion of generalized 3-Sasakian structure.

Our purpose in this paper is to study the notion of QR-submanifolds on a manifold endowed with a generalized 3-Sasakian structure. More exactly we have obtained a classification of a totally contact umbilical QR-submanifolds of a manifold with a 3-Sasakian structure.

1 Preliminaries

Let $\tilde{M}$ be a $(4n+3)$-dimensional differentiable manifold with an almost contact metric 3-structure (f_a, ξ_a, η_a, g), $a \in \{1,2,3\}$, that satisfy the relations

$$\begin{aligned} &a)\ f_a^2 = -I + \eta_a \otimes \xi_a, \quad b)\ \eta_a(\xi_b) = \delta_{ab} \quad c)\ f_a(\xi_b) = -f_b(\xi_a) = \xi_c, \\ &d)\ \eta_a \circ f_b = -\eta_b \circ f_a = \eta_c, \quad e)\ f_a \circ f_b - \eta_b \otimes \xi_a = -f_b \circ f_a + \eta_a \otimes \xi_b = f_c, \\ &f)\ \eta_a(X) = g(X, \xi_a), \quad g)\ g(f_a X, f_a Y) = g(X,Y) - \eta_a(X)\eta_a(Y), \end{aligned} \tag{1.1}$$

for any cyclic permutation (a,b,c) of $(1,2,3)$, where X and Y are the vector fields tangent to $\tilde{M}$, δ is the Kronecker's delta. The notion of 3-Sasakian structure has been introduced by Kuo in [7] and was generalized by Bejancu-Faran in [5]. More exactly, the manifold $\tilde{M}$ is a manifold with a generalized 3-Sasakian structure, if there exists the local 1-forms α_{ab} so that $\alpha_{ab} + \alpha_{ba} = 0$ and

$$(\tilde{\nabla}_X f_a)Y = g(X,Y)\xi_a - \eta_a(Y)X + \alpha_{ab}(X) f_b(Y) + \alpha_{ac}(X) f_c(Y), \tag{1.2}$$

for any vector fields X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$ and $\{a,b,c\}$ is a cyclic permutation of $\{1,2,3\}$.

Throughout the paper, all manifolds and maps are supposed differentiable of class C^∞. We denote by $F(M)$ the module of the differentiable functions on $\tilde{M}$ and by $\Gamma(E)$ the module of smooth sections of a vector bundle E over $\tilde{M}$. We use the same notations for any manifolds involved in the study.

By straightforward calculation using (1.1b)-(1.1d) one deduces that

$$\tilde{\nabla}_X \xi_a = -f_a X + \alpha_{ab}(X)\xi_b + \alpha_{ac}(X)\xi_c, \quad \forall X \in \Gamma(T\tilde{M}). \tag{1.3}$$

The curvature tensor K of $\tilde{M}$ is defined by

$$K(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z, \quad \forall X,Y,Z \in \Gamma(T\tilde{M}). \tag{1.4}$$

Theorem 1.1. *Let $\tilde{M}$ be a $4n+3$-dimensional differentiable manifold with a generalized 3-Sasakian structure. Then the curvature tensor field K verifies*

$$\begin{aligned} a)\ &\tilde{K}(X,Y)f_a Z = f_a \tilde{K}(X,Y)Z + g(f_a X, Z)Y - g(Y,Z) f_a X + \\ &+ g(X,Z) f_a Y - g(f_a Y, Z)X + A_{ab}(X,Y) f_b Z + A_{ac}(X,Y) f_c Z, \\ b)\ &g(\tilde{K}(X,Y) f_a Z, f_a W) = g(\tilde{K}(X,Y)Z, W) - g(Y,Z) g(X,W) + \\ &+ g(f_a X, W) g(f_a Y, Z) - g(f_a X, Z) g(f_a Y, W) + g(Y,W) g(X,Z) + \\ &+ A_{ab}(X,Y) g(Z, f_c W) - A_{ac}(X,Y) g(Z, f_b W), \\ c)\ &\tilde{K}(X,Y)\xi_a = \eta_a(Y)X - \eta_a(X)Y + A_{ab}(X,Y)\xi_b + A_{ac}(X,Y)\xi_c, \\ &\forall X,Y,Z,W \in \Gamma(T\tilde{M}), \end{aligned} \tag{1.5}$$

where $A_{ab} = 2d_{ab} + \alpha_{ac} \wedge \alpha_{bc}$, are the local 2-form on $\tilde{M}$.

Proof. The assertion (1.5a) follows by using (1.2)-(1.4) and the Ricci identity

$$\tilde{K}(X,Y)f_a - f_a\tilde{K}(X,Y) = \tilde{\nabla}_X\tilde{\nabla}_Y f_a - \tilde{\nabla}_Y\tilde{\nabla}_X f_a - \tilde{\nabla}_{[X,Y]}f_a, \quad X,Y \in \Gamma(T\tilde{M}).$$

One deduces the assertions (1.5b) and (1.5c) from (1.5a) and (1.2), (1.3) by direct calculation.

Next let $\{e_j, e_{an+j} = f_a e_j, \xi_a = e_{4n+a},\}$ $j = 1,\dots,n$ $a = 1,2,3$, be an orthonormal field of frame on $\tilde{M}$. Then the Ricci tensor S on $\tilde{M}$ is given by

$$S(X,Y) = \sum_{i=1}^{4n+3} g(\tilde{K}(e_i,X)Y,e_i), \quad X,Y \in \Gamma(T\tilde{M}). \tag{1.6}$$

Proposition 1.1. *The 2-forms A_{ab} from the Theorem 1.1 verify the next relations*

$$\begin{aligned} &a)\ \sum_{i=1}^{4n+3} g(\tilde{K}(X,Y)e_i, f_a e_i) = 2g(X, f_a Y) + 2(n+1)A_{bc}(X,Y),\\ &b)\ A_{ab}(X, f_c Y) = A_{ca}(X, f_b Y) = A_{bc}(X, f_a) = \tfrac{(4n+2)g(X,Y) - S(X,Y)}{n+3}, \ n > 0,\\ &c)\ A_{bc}(X, f_a Y) + A_{bc}(f_a X, Y) = 0, \quad d)\ A_{ab}(X,\xi_a) = 0, \quad \forall X,Y \in \Gamma(T\tilde{M}). \end{aligned} \tag{1.7}$$

Proof. If we put $X = W = e_i$, $i = 1,\dots,4n+3$ in (1.5b), by using (1.1g), (1.6) we deduce

$$\begin{aligned} \sum_{i=1}^{4n+3} g(\tilde{K}(e_i,Y)f_a Z, f_a e_i) = S(Y,Z) - (4n+2)g(f_a Y, f_a Z) + \\ + A_{ab}(Y, f_c Z) - A_{ac}(Y, f_b Z), \quad \forall X,Y,Z \in \Gamma(T\tilde{M}) \end{aligned} \tag{1.8}$$

On the other hand, if we consider $Z = e_i$, $W = f_b e_i$ in (1.5b) we obtain

$$\begin{aligned} &\sum_{i=1}^{4n+3} (g(\tilde{K}(X,Y)f_a e_i, f_b f_a e_i) - g(\tilde{K}(X,Y)e_i, f_b e_i)) = \\ &= (4n+2)A_{ca}(X,Y) - 4g(x, f_b Y) + 2\eta_a(X)\eta_c(Y) - 2\eta_a(Y)\eta_c(X), \quad \forall X,Y \in \Gamma(T\tilde{M}). \end{aligned} \tag{1.9}$$

By straightforward calculation, using (1.1b), (1.1e), we infer that

$$\begin{aligned} &\sum_{i=1}^{4n+3} (g(\tilde{K}(X,Y)f_a e_i, f_a f_b e_i) - g(\tilde{K}(X,Y)e_i, f_b e_i)) = \\ &= \sum_{i=1}^{4n+3} (g(\tilde{K}(X,Y)f_a e_i, f_c e_i) - g(\tilde{K}(X,Y)e_i, f_b e_i)) + g(\tilde{K}(X,Y)\xi_c, \xi_a) = \\ &= -2\sum_{i=1}^{4n+3} g(\tilde{K}(X,Y)e_i, f_b e_i) - 2g(\tilde{K}(X,Y)\xi_a, \xi_c). \end{aligned} \tag{1.10}$$

Therefore from (1.5c), (1.9) and (1.10), we obtain (1,7a). Next, by using the first Bianchi identity we infer that

$$\sum_{i=1}^{4n+3} g(\tilde{K}(e_i,Y)f_aZ,f_ae_i) = -\sum_{i=1}^{4n+3}(g(\tilde{K}(Y,f_aZ)e_i,f_ae_i)+g(\tilde{K}(f_aZ,e_i)Y,f_ae_i)) =$$
$$= -2\sum_{i=1}^{4n+3} g(\tilde{K}(Y,f_aZ)e_i,f_ae_i), \quad \forall Y,Z \in \Gamma(T\tilde{M}). \tag{1.11}$$

The relations (1.8)-(1.11) imply

$$S(Y,Z) = (4n+2)g(Y,Z) - (n+1)A_{bc}(Y,f_aZ)- \\ -A_{ca}(Y,f_bZ) - A_{ab}(Y,f_cZ), \quad \forall Y,Z \in \Gamma(T\tilde{M}). \tag{1.12}$$

Thus, if $n > 0$, from (1.12) we infer that

$$S(X,Y) = (4n+2)g(X,Y) - (n+3)A_{ab}(X,f_cZ), \quad \forall X,Y \in \Gamma(T\tilde{M}), \tag{1.13}$$

for any cyclic permutation (a,b,c) of (1,2,3), that prove the relations (1.7b). Next taking into account the properties of the tensors S, g and that A_{ab} is a 2-form, we obtain (1.7d). If we put $Y = \xi_a$ in (1.7d), by using (1.1c) we deduce (1.7c). The proof is complete.

By direct calculation, by using (1.1g), (1.7c) and (1.13) we deduce

Proposition 1.2. *Let $\tilde{M}$ be a $(4n+3)$-dimensional differentiable manifold endowed with a generalized 3-Sasakian structure. Then the Ricci tensor S satisfies*

$$S(f_aX,f_aY) = S(X,Y) - (4n+2)\eta_a(X)\eta_a(Y), \quad \forall X,Y \in \Gamma(T\tilde{M})$$

2 Basic properties

In this paragraph we prove some important properties of some tensor field involved in the study. First we have

Lemma 2.1. *If $\tilde{M}$ is an almost contact metric manifold endowed with a generalized 3-Sasakian structure, then there the next relations exist*

$$\begin{aligned} &a)\ \ A_{ab}(X,\xi_c) = 0, \\ &b)\ \ A_{bc}(X,Y) = A_{ba}(X,f_bY) = A_{ca}(X,f_cY), \quad \forall X,Y \in \Gamma(T\tilde{M}). \end{aligned} \tag{2.1}$$

Proof. Let us consider f_bX instead of X, then from (1.7c), by using (1.1c) and (1.7d) we obtain the relation (2.1a). The relation (2.1b) is obtained from (1.7b) if we put f_aY instead of Y, by using (1.1e) and (2.1a).

Theorem 2.1. *Let $\tilde{M}$ be an almost contact metric structure endowed with a generalized*

3-Sasakian structure. Then the Ricci tensor S verifies

$$\begin{aligned}
&a)\quad (\tilde{\nabla}_X S)(Y, f_a Z) + (\tilde{\nabla}_Y S)(Z, f_a X) + (\tilde{\nabla}_Z)(X, f_a Y) = \\
&\quad = -(n+3)(\eta_a(X) A_{bc}(Y, f_a Z) + \eta_a(Y) A_{bc}(Z, f_a X) + \eta_a(Z) A_{bc}(X, f_a Y)), \\
&b)\quad (\tilde{\nabla}_X S)(Y, \xi_a) = (n+3) A_{bc}(Y, X), \\
&c)\quad (\tilde{\nabla}_{\xi_a} S)(Y, Z) = 0, \quad \forall X, Y, Z \in \Gamma(T\tilde{M}).
\end{aligned} \tag{2.2}$$

Proof. Let $X, Y, Z \in \Gamma(T\tilde{M})$. By using (1.2), (1.3), (1.14a) and (1.14b) it is obtained

$$\begin{aligned}
&(\tilde{\nabla}_X S)(f_a X, f_a Z) = (\tilde{\nabla}_X S)(Y, Z) - (4n+2)(\eta_a(Y)(\tilde{\nabla}_X \eta_a) Z - \\
&-\eta_a(Z)(\tilde{\nabla}_X \eta_a) Y) - S(f_a Y, (\tilde{\nabla}_X f_a) Z) - S((\tilde{\nabla}_X f_a) Y, f_a Z) = \\
&= (\tilde{\nabla}_X S)(Y, Z) - (n+3)(\eta_a(Y) A_{bc}(X, Z) + \eta_a(Z) A_{bc}(X, Y))
\end{aligned} \tag{2.3}$$

Next, by using (1.1e), (1.2), (1.7b) and (1.7d) we deduce that

$$\begin{aligned}
(\tilde{\nabla}_X S)(Y, f_a Z) &= (4n+2) g(Y, (\tilde{\nabla}_X f_a) Z) + (n+3)(\tilde{\nabla}_X A_{bc})(Y, Z) - S(Y, (\tilde{\nabla}_X f_a) Z) = \\
&= (n+3)((\tilde{\nabla}_X A_{bc})(Y, Z) + A_{bc}(Y, f_a((\tilde{\nabla}_X f_a) Z)) = \\
&= (n+3)(\tilde{\nabla}_X A_{bc})(Y, Z) - \eta_a(Z) A_{bc}(X, f_a Y) + \alpha_{ab}(X) A_{ac}(Y, Z) - \alpha_{ac}(X) A_{ab}(Y, Z)).
\end{aligned}$$

Now, from the above relation we get that

$$\begin{aligned}
&(\tilde{\nabla}_X S)(Y, f_a Z) + (\tilde{\nabla}_Y S)(Z, f_a X) + (\tilde{\nabla}_Z S)(X, f_a Y) = \\
&= (n+3)(dA_{bc} + \alpha_{ab} \wedge A_{ac} - \alpha_{ac} \wedge A_{ab})(X, Y, Z) - (n+3)(\eta_a(X) A_{bc}(Y, f_a Z) + \\
&+\eta_a(Y) A_{bc}(Z, f_a X) + \eta_a(Z) A_{bc}(X, f_a Y)),
\end{aligned}$$

which prove the relation a).

Next, by using (1.3), (1.7a), (1.14a), (1.17) and (2.1a) we infer

$$\begin{aligned}
(\tilde{\nabla}_X S)(Y, \xi_a) &= (4n+2)(\tilde{\nabla}_X \eta_a) Y - S(Y, \tilde{\nabla}_X \xi_a) = (4n+2) g(Y, \tilde{\nabla}_X \xi_a) - S(Y, \tilde{\nabla}_X \xi_a) = \\
&= (n+3) A_{bc}(Y, f_a \tilde{\nabla}_X \xi_a) = (n+3) A_{bc}(Y, X),
\end{aligned}$$

and (2.2b) is proved.

Now if we consider ξ_a instead of X and $f_a Z$ instead of Z, the relation (2.2a) implies

$$(\tilde{\nabla}_{\xi_a} S)(Y, f_a^2 Z) + (\tilde{\nabla}_{f_a Z} S)(\xi_a, f_a Y) = -(n+3) A_{bc}(Y, f_a^2 Z).$$

Taking into account (1.7c), (1.14a) and (2.2b), from the above relation we obtain (2.2c). The proof is complete.

Theorem 2.2. *If $\tilde{M}$ is a manifold endowed with a generalized 3-Sasakian structure, then $\tilde{M}$ is η-parallel manifold, that is $(\tilde{\nabla}_X S)(f_c Y, f_a Z) = 0$.*

Proof. First, taking into account (2.2c), we prove the theorem for $X, Y, Z \in \Gamma(T\tilde{M})$ with $\eta_a(X) = \eta_a(Y) = \eta_a(Z) = 0$. Now, we consider $f_a Z$ instead of Z and the relation (2.2a) implies

$$(\tilde{\nabla}_Y S)(f_a Z, f_a Y) - (\tilde{\nabla}_X S)(Y, Z) + (\tilde{\nabla}_{f_a Z} S)(X, f_a Y) = 0.$$

By using (1.1e) and (2.3), we deduce that

$$\begin{aligned}(\tilde{\nabla}_X S)(Y, Z) - (\tilde{\nabla}_Y S)(X, Z) = (\tilde{\nabla}_{f_a Z} S)(X, f_a Y) = \\ = (\tilde{\nabla}_{f_a Z} S)(f_b X, f_b f_a Y) = -(\tilde{\nabla}_{f_a Z} S)(f_b Y, f_c Z).\end{aligned} \tag{2.4}$$

Now from (2.4) we infer that

$$(\tilde{\nabla}_{f_a Z} S)(f_b X, f_c Y) = (\tilde{\nabla}_{f_b Z} S)(f_c X, f_a Y) = (\tilde{\nabla}_{f_c Z} S)(f_a X, f_b Y). \tag{2.5}$$

Thus, if we consider $f_b X, f_c Y, f_a Z$ instead of X, Y, Z, respectively, by using (1.1a), (1.1c), from (2.4) we obtain

$$(\tilde{\nabla}_Z)(X, Y) = (\tilde{\nabla}_{f_c Z} S)(f_a X, f_b Y) = -(\tilde{\nabla}_{f_b Z}(f_c X, f_a Y). \tag{2.6}$$

Therefore from (2.2c), (2.5) and (2.6) we infer the relation desired.

3 QR-submanifolds

Next, let M be a m-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$, and suppose that the structure vector fields ξ_1, ξ_2, ξ_3 of $\tilde{M}$ are tangent to M, respectively. We denote by TM and $TM^\perp$ the tangent bundle and the normal bundle to M, respectively. We also denote by $\{\xi\}$ the distribution spanned by ξ_1, ξ_2, ξ_3 on M. The induced metric tensor on M will be denoted by the same symbol g.

According to A.Bejancu [3], we say that the submanifold M of a manifold with a generalized 3-Sasakian structure is a QR-submanifold, if there exists a vector subbundle ν of $TM^\perp$ such as

$$f_a(\nu) = \nu; \quad f_a(\nu^\perp) \subseteq TM, \quad a \in \{1, 2, 3\},$$

where $\nu^\perp$ is the complementary orthogonal bundle to ν in $TM^\perp$. If in particular $\nu = TM^\perp$ or $\nu = \{0\}$, we say that M becomes a quaternionic submanifold (see Chen [8]) or anti-quaternionic submanifold (see Pak [9]). It is easy to see that any real hypersurface of $\tilde{M}$ is a QR-submanifold. Suppose that M is a QR-submanifold which is not a quaternionic submanifold. Next, denote $f_a(\nu_x^\perp)$ by D_{ax}, $a \in \{1, 2, 3\}$ $x \in M$, $a \in \{1, 2, 3\}$. It is easy to see that D_{1x}, D_{2x}, D_{3x} are mutually orthogonal subspaces of $T_x M$ and have the same dimensions as the dimension of $\nu_x^\perp$. We note that the subspaces D_{ax}, $a \in \{1, 2, 3\}$ do not define in general a distribution on M, but the mapping

$$D^\perp : x \to D_x^\perp = D_{1x} \oplus D_{2x} \oplus D_{3x}, \quad x \in M,$$

is a $3s$-dimensional distribution on M ($s = \dim \nu_x^\perp$). By straightforward calculation it is easy to see that

$$a)\quad f_a(D_{ax}) = \nu_x^{\perp};\qquad b)\quad f_a(D_{bx}) = D_{cx}, \tag{3.1}$$

for each $x \in M$, and (a,b,c) there is a cyclic permutation of $(1,2,3)$. We denote by D the complementary orthogonal distribution to $D^{\perp} \oplus \{\xi\}$ in TM. It follows that the distribution D is invariant with respect to the action of f_1, f_2, f_3, that is $f_a(D) = D, a \in \{1,2,3\}$. Thus M is a QR-submanifold of a manifold $\tilde{M}$ with a Sasakian 3-structure if

$$TM = D \oplus D^{\perp} \oplus \{\xi\}, \tag{3.2}$$

where D, $\{\xi\}$ and $D^{\perp}$ are the above distributions. We note that $D^{\perp}$ is not anti-invariant distribution (see (3.1b)).

From the general theory of Riemannian submanifolds, we recall the Gauss and Weingarten formulae

$$\begin{aligned} &a)\quad \tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),\\ &b)\quad \tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N,\ \ \forall X,Y \in \Gamma(TM),\ N \in \Gamma(TM^{\perp}), \end{aligned} \tag{3.3}$$

where h is the second fundamental form of M, A_N is the shape operator with respect to the normal section N, ∇ and $\nabla^{\perp}$ are the induced connections by $\tilde{\nabla}$ on TM and $TM^{\perp}$, respectively. The Codazzi equation is given by

$$\begin{aligned} g(K(X,Y)Z,N) &= g((\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),N),\\ &\forall X,Y,Z \in \Gamma(TM),\ N \in \Gamma(TM^{\perp}). \end{aligned} \tag{3.4}$$

It is known that if $\{e_i\}$ $i = 1,\dots,m$ is an orthonormal basis of $\Gamma(TM)$, then the mean curvature vector field of M, dented by H, is given by

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i,e_i).$$

The submanifold M is called totally contact umbilical if the second fundamental form h of M is expressed as

$$h(X,Y) = \sum_{a=1}^{3}(g(f_aX,f_aY)H + \eta_a(X)h(Y,\xi_a) + \eta_a(Y)h(X,\xi_a)),\quad \forall X,Y \in \Gamma(TM). \tag{3.5}$$

If $H = 0$ and (3.5) hold, then M is called totally contact geodesic submanifold of $\tilde{M}$.

We recall that M is an extrinsic sphere of $\tilde{M}$ if it is a totally contact umbilical and has parallel the mean curvature vector field $H \neq 0$, that is,

$$\nabla_X^{\perp} H = 0,\quad \forall X \in \Gamma(TM).$$

By straightforward calculation, using (1.3), (3.1), (3.2) and (3.3a) we can prove

Proposition 3.1. *Let M be a QR-submanifold of a manifold $\tilde{M}$ with a generalized 3-Sasakian structure. Then we have*

$$a)\ h(X,\xi_a)=0;\quad b)\ h(Z,\xi_a)=-f_aZ,\ \forall X\in\Gamma(D),\ Z\in\Gamma(f_a(\nu^{\perp})). \tag{3.6}$$

4 Main results.

Our purpose in this paragraph is to study some properties of a totally contact umbilical QR-submanifold M of a manifold $\tilde{M}$ with a generalized 3-Sasakian structure. More exactly we prove that if M is a totally contact umbilical QR-proper submanifold ($\dim D > 0$; $\dim D^{\perp} > 0$), then M must be totally contact geodesic if $s \neq 1$ or M is an extrinsic sphere if $s = 1$. To this end we first prove the following general lemma

Lemma 4.1. *Let M be a totally contact umbilical QR-submanifold of a manifold $\tilde{M}$ endowed with a generalized 3-Sasakian structure and $D \neq \{0\}$. Then the mean curvature vector field H of M is a globally section of $\Gamma(\nu^{\perp})$. Moreover, if $s \neq 1$, then M is totally contact geodesic.*

Proof. Let $X \in \Gamma(D)$ a unit vector field and $N \in \Gamma(\nu)$. By using (1.1g), (1.2), (3.3) and (3.5) we deduce that

$$g(H,N)=g(g(X,X)H,N)=g(\tilde{\nabla}_X X,N)=g(\tilde{\nabla}_X f_aX-(\tilde{\nabla}_X f_a)X,f_aN)=$$
$$=g(h(X,f_aX),f_aN)=g(X,f_aX)g(H,f_aN)=0.$$

Thus from the above relation we deduce that $H \in \Gamma(\nu)$. Therefore if $s = 0$, then $H = 0$ and M is totally contact geodesic. Next we suppose $s > 1$, and let $Y, Z \in \Gamma(f_a(\nu^{\perp}))$ so that $g(Z,Z) = 1$ and $g(Y,Z) = 0$. Then by using (1.1g), (1.2), (3.3) and (3.5) we get that

$$g(H,f_aY)=g(\tilde{\nabla}_Z Z,f_aY)=g((\tilde{\nabla}_Z f_a)Z-\tilde{\nabla}_Z f_aZ,Y)=$$
$$=g(f_aZ,\tilde{\nabla}_Z Y)=g(f_aZ,h(Y,Z))=0,$$

which implies that $H \in \Gamma(\nu)$. Finally, if $s > 1$ we obtain $H = 0$, that means M is totally contact geodesic. Next we deal with the remaining case $s = 1$ and $H \neq 0$.

Lemma 4.2. *Let M be a totally contact umbilical QR-submanifold of a manifold $\tilde{M}$ with a generalized 3-Sasakian structure. Then we have*

$$\nabla^{\perp}_X H \in \Gamma(\nu^{\perp}),\quad \forall X \in \Gamma(TM).$$

Proof. Let $X \in \Gamma(TM)$ and $N \in \Gamma(\nu)$. Now from Lemma 3.1 we have $H \in \Gamma(\nu^{\perp})$. Next, by using (1.1g), (1.2), (3.3a) and (3.5) we infer that

$$g(\nabla^{\perp}_X H,N)=g(\tilde{\nabla}_X f_aH-(\tilde{\nabla}_X f_a)H,f_aN)=g(h(X,f_aH),f_aN)=$$
$$=g(X,f_aH)g(H,f_aN)=0.$$

Therefore our assertion is proved. Next we prove the main result of the paper

Theorem 4.1. *Let M be a proper totally contact umbilical QR-submanifold of a manifold $\tilde{M}$ endowed with a generalized 3-Sasakian structure, such that $\dim \nu_x^{\perp} = 1$, for any $x \in M$ and $H \neq 0$. Then M is an extrinsic sphere.*

Proof. Because $H \neq 0$ and M is supposed to be connected, denote by U the unit vector field so that $H = g(H,H)U$ and let $W_a = f_a U$. Next let $X, Y \in \Gamma(D)$ and by using (1.5a) we get that

$$\begin{aligned} g(\tilde{K}(W_1,X)f_1Y,U) &= g(f_1\tilde{K}(W_1,X)Y + g(X,Y)U,U) = \\ &= g(X,Y) - g(\tilde{K}(W_1,X)Y,W_1). \end{aligned} \tag{4.1}$$

On the other hand, by using (3.4), (3.5) and (3.6a) we deduce that

$$\begin{aligned} &g(\tilde{K}(W_1,X)f_1Y,U) = g((\nabla_{W_1}h)(X,f_1Y) - (\nabla_X h)(W_1,f_1Y),U) = \\ &= 3g(X,f_1Y)g(\nabla^{\perp}_{W_1}H,U) - 3g(W_1,f_1Y)g(\nabla^{\perp}_X H,U) = 3g(X,f_1Y)g(\nabla^{\perp}_{W_1}H,U). \end{aligned} \tag{4.2}$$

The relations (4.1) and (4.2) imply

$$g(X,Y) - g(\tilde{K}(W_1,X)Y,W_1) = 3g(X,f_1Y)g(\nabla^{\perp}_{W_1}H,U).$$

But, using the symmetries of the tensors g and K, and using (1.1g) we get $g(\nabla^{\perp}_{W_1}H,U) = 0$ and together with the result of Lemma 3.2, we obtain $\nabla^{\perp}_Z H = 0$, $Z \in \Gamma(D^{\perp})$. Next, let $X \in \Gamma(D)$ be a unit vector field. By using (3.4), (3.5) and (3.6a) we infer that

$$g(\tilde{K}(\xi_1,X)X,U) = g((\nabla_{\xi_1}h)(X,X) - (\nabla_X h)(\xi_1,X),U) = g(\nabla^{\perp}_{\xi_1}H,U).$$

Taking into account (1.5c), the fact that $U \in \Gamma(\nu^{\perp})$ and Lemma 3.2 we obtain $\nabla^{\perp}_{\xi_1}H = 0$. In the same way we can prove that $\nabla^{\perp}_Y H = 0$, $\forall Y \in \Gamma(\{\xi\})$. Finally we consider $X \in \Gamma(D)$ a unit vector field and by using (3.4), (3.5) and (3.6a) we get that

$$\begin{aligned} g(\tilde{K}(f_1X,f_2X)f_3X,U) &= g((\nabla_{f_1X}h)(f_2X,f_3X) - (\nabla_{f_2X}h)(f_1X,f_3X),U) = \\ &= 3g(f_2X,f_3X)g(\nabla^{\perp}_{f_1X}H,U) - 3g(f_1X,f_3X)g(\nabla^{\perp}_{f_2X}H,U) = 0. \end{aligned} \tag{4.3}$$

On the other hand, by using (1.5a) and (1.7d) we obtain that

$$\begin{aligned} g(\tilde{K}(f_1X,f_2X)f_3X,U) &= g(\tilde{K}(f_3X,U)f_1X,f_2X) = \\ &= -g(\tilde{K}(f_3X,U)X,f_3X) = -g(\tilde{K}(X,f_3X)f_3X,U). \end{aligned} \tag{4.4}$$

By direct calculation one deduces

$$\begin{aligned} g(\tilde{K}(X,f_3X)f_3X,U) &= g((\nabla_X h)(f_3X,f_3X) - (\nabla_{f_3X}h)(X,f_3X),U) = \\ &= 3g(\nabla^{\perp}_X H,U), \ \forall X \in \Gamma(D). \end{aligned} \tag{4.5}$$

The relations (4.3)-(4.5) and Lemma 3.2 prove that $\nabla^{\perp}_X H = 0$, $\forall X \in \Gamma(D)$. Finally we have proved that $\nabla^{\perp}_X H = 0$, $\forall X \in \Gamma(TM)$. The proof is complete.

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