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On Boundary Behavior of Quasiconformal Mappings in $\overline{R}^n$

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Abstract. In this note we will give a boundary extension theorem for quasiconformal mappings on globally quasiconformally collared domains. We will show that quasiextremal distance (Sobolev capacity) domain has property P_2^* at each (finite) boundary point and will provide some boundary extension theorems for quasiconformal mapping where one of the domains in question is QED or SC.

All the sets considered in this note are assumed to lie in $\overline{R}^n = R^n \cup \{\infty\}$, $n \geq 2$. Given a point $x \in R^n$ and a number $r > 0$, we let $B^n(x,r)$ denote the n-dimensional ball and $S^{n-1}(x,r)$ its $(n-1)$-dimensional boundary sphere. We will also employ the notations $B^n = B^n(0,1)$, $S^{n-1} = S^{n-1}(0,1)$.

Mention must be made of the fact that all neighborhood considered in this Note, are supposed to be open; in the same time, we precise that the sets D, D' are domains in $\overline{R}^n$.

Let f be a mapping of a domain D into $\overline{R}^n$ and let b be a point in ∂D. The cluster set $C(f,b)$ of f at b is the set of all points $b' \in \overline{R}^n$ for which there exists a sequence (b_k) in D such that $b_k \to b$ and $f(b_k) \to b'$. Obviously, $C(f,b)$ is a non-empty compact set (see [6]), there exists the limit of f at b if and only if $C(f,b)$ has a single point, and $C(f,b) \subset \partial f(D)$ if f is a homeomorphism.

If Γ is a family of paths in $\overline{R}^n$ we denote by $F(\Gamma)$ the family of all Borel functions $\rho : \overline{R}^n \to [0,\infty]$ for which $\int_\gamma \rho ds \geq 1$ for every rectifiable path $\gamma \in \Gamma$. The modulus of Γ is defined as $M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\overline{R}^n} \rho^n dm$.

The conformal capacity is definited by $cap(E,F;D) = \inf_{u \in L} \int_D |\nabla u|^n$, where $E; F \subset \overline{D}$ are disjoint, non-empty, compact sets and the infimum is taken over all functions in the class $L = L(E,F;D) = \{u \in L_n^1(D) \cap C(D \cup E \cup F); u/_E, u/_F \geq 1\}$. Here $L_n^1(D)$ denotes the Sobolev space of locally integrable functions $u : D \to R \cup \{\pm\infty\}$ which satisfy $\int_D |\nabla u|^n < \infty$, where ∇u represents the distributional gradient of u and $C(D \cup E \cup F)$ is the family of all continuous functions $u : D \cup E \cup F \to R \cup \{\pm\infty\}$.

If $E, F, G \subset \overline{R}^n$, $E \subset \overline{G}$, $F \subset \overline{G}$, we denote by $\Delta(E,F;G)$ the family of all paths which join E and F in G.

A homeomorphism $f : D \to D'$ is said to be K-quasiconformal, $1 \leq K < \infty$, if it satisfies the double inequality:

$$\frac{1}{K}M(\Gamma) \leq M(f(\Gamma)) \leq KM(\Gamma)$$

for each path family Γ in D. Here $f(\Gamma) = \{f \circ \alpha,\ \alpha \in \Gamma\}$.

A homeomorphism f is said to be quasiconformal if it is K-quasiconformal for some K. The maximal dilatation of f, denoted by $K(f)$, is then defined as the least K for which f is K-quasiconformal.

Definition 1. *A domain D in $\overline{R}^n$ is globally quasiconformally collared on the boundary if there exists a neighborhood U of ∂D and a homeomorphism*

$$g : U \cap \overline{D} \to \{x \in R^n,\ a < |x| \leq 1\}, \quad a \geq 0,$$

such that $g/_{U\cap D}$ is quasiconformal.

Gehring introduced above definition and proved the next theorem.

Theorem 1. *A global quasiconformal collared on the boundary domain in $\overline{R}^n$ is quasiconformally echivalent to B^n.*

Now, we will prove some theorems by whose aid we will give a necessary and sufficient condition of extension at boundary for quasiconformal mappings.

Theorem 2. *If $D \subset \overline{R}^n$ is a globally quasiconformally collared on the boundary, then D is locally connected on the boundary.*

Proof. Since D is globally quasiconformally collared on the boundary, there exists a neighborhood U of ∂D and a homeomorphism $g : U \cap \overline{D} \to \{x \in R^n, a < |x| \leq 1\}$, $a \geq 0$, such that $g/_{U\cap D}$ is quasiconformal. Let b be a boundary point of D and let V be a neighborhood of b. Obviously, $U_1 = V \cap U$ is a neighborhood of b. Set $C = \{x \in R^n,\ a < |x| < 1\}$ and hence $C \cup S^{n-1} = \{x \in R^n, a < |x| \leq 1\}$. We can find $0 < r < 1 - a$ such that $g^{-1}(B^n(g(b), r) \cap (C \cup S^{n-1})) \subset U_1 \cap \overline{D}$. Then $W = g^{-1}(B^n(g(b), r) \cap (C \cup S^{n-1})) \cup (U_1 - \overline{D})$ is a neighborhood of b. Moreover, $W \subset (U_1 \cap \overline{D}) \cup (U_1 - \overline{D}) \subset U_1 \subset V$. On the other hand, $W \cap D = g^{-1}(B^n(g(b), r) \cap C)$ is a connected set. It follows that D is locally connected at b.

Theorem 3. *If D, $D^{'}$ are domains in $\overline{R}^n$, globally quasiconformally collared on the boundary, then each quasiconformal mapping $f : D \to D^{'}$ can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}^{'}$.*

Proof. Since D, $D^{'}$ are globally quasiconformally collared on the boundary, by Theorem 2, it follows that D, $D^{'}$ are locally connected on the boundary and using Theorem 1, D, $D^{'}$ are quasiconformally echivalent to B^n. On the other hand, B^n is locally quasiconformally collared on the boundary and applying Theorem 17.18. [7], we obtain the desirable conclusion.

Theorem 4. *Let $f : D \to D^{'}$ be a quasiconformal mapping which has a homeomorphic extension to $\overline{D} \to \overline{D}^{'}$. If D is globally quasiconformally collared on the boundary, then $D^{'}$ is globally quasiconformally collared on the boundary.*

Proof. Since D is globally quasiconformally collared on the boundary, there exists a neighborhood U of ∂D and a homeomorphism $g : U \cap \overline{D} \to \{x \in R^n,\ a < |x| \leq 1\}$, $a \geq 0$ such that $g/_{U \cap D}$ is quasiconformal. Choose V' a neighborhood of $\partial D'$ such that $f^{-1}(V' \cap \overline{D}') \subset U \cap \overline{D}$. Then $U' = V' \cup f(U \cap \overline{D})$ is a neighborhood of $\partial D'$ and $U' \cap \overline{D}' = (V' \cap \overline{D}') \cup f(U \cap \overline{D}) = f(U \cap \overline{D})$. Setting $h = g \circ f^{-1}$ we obtain a homeomorphism $h : U' \cap \overline{D}' \to \{x \in R^n,\ a < |x| \leq 1\}$ such that $h/_{U' \cap D'}$ is quasiconformal.

Thus D' is globally quasiconformally collared on the boundary.

By theorems 3, 4 we obtain the theorem:

Theorem 5. *Let $f : D \to D'$ be a quasiconformal mapping. If D is globally quasiconformally collared on the boundary, then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}'$ if and only if D' is globally quasiconformally collared on the boundary.*

Next, we give some definitions and theorems necessary to study the behavior at boundary of a quasiconformal mapping, where one of the domains in question is QED or SC.

Definition 2. *D has property P_1 at a boundary point b if the following condition is satisfied: $M(\Delta(E, F; D)) = \infty$, whenever E, F are connected subsets of D with $b \in \overline{E} \cap \overline{F}$.*

Definition 3. *D has property P_2^* at b if the following condition is satisfied:*

for each point $b_1 \in \partial D$, $b_1 \neq b$, there exists a continuum $F \subset D$ and a constant $\delta > 0$ such that $M(\Delta(E, F; D)) \geq \delta$, whenever E is a connected set in D such that $b, b_1 \in \overline{E}$.

This property P_2^* is different by property P_2 considerate in [5], Definition 17.5(4). Definition 2 is the same like Definition 17.5(3) [7]. We use the notation $\Delta(E, F) = \Delta(E, F; D)$ when $D = R^n$ or $D = \overline{R}^n$. Gehring and Martio [3] introduced the class of quasiextremal distance domains.

Definition 4. *A quasiextremal distance domain $D \subset \overline{R}^n$ is a domain which satisfies the following condition : there exists $k \geq 0$ such that*

$$M(\Delta(E, F)) \leq k M(\Delta(E, F; D))$$

for each pair of disjoint continua $E, F \subset D$. We abbreviate this situation by saying that D is QED.

Definition 5. *Sobolev capacity of E, F relative to D is defined by*

$$s - cap(E, F; D) = \inf_{u \in W} \int_D (|u|^n + |\nabla u|^n)$$

where $E, F \subset \overline{D}$ are disjoint, non-empty, compact sets and the infimum is taken over all functions in the class

$$W = W(E, F; D) = \{u \in W_n^1(D) \cap C(D \cup E \cup F);\ u/_E \leq c,\ u/_F \geq c + 1,\ c \in R\}$$

and $W_n^1(D) = L_n^1(D) \cap L^n(D)$.

Definition 6. *A Sobolev capacity domain $D \subset R^n$ is a domain which satisfies the following* *condition : there exists* $M \geq 0$ *such that*

$$s - cap(E, F) \leq M \cdot s - cap(E, F; D)$$

for each pair of disjoint continua $E, F \subset D$. *We abbreviate this situation by saying that* D *is SC.*

Pekka Koskela [5] introduced this class of Sobolev capacity domains.

Remark 1. The Sobolev capacity always dominates the conformal capacity (see [5]). One fundamental difference between QED and SC domains is that, unlike the former SC domains are not invariant with respect to Mobius transformations. In fact, SC domains are not even affine invariant.

Remark 2. Gehring and Martio ([3], 2.11) verified that QED domains are linearly locally connected domains and hence locally connected at each boundary point ([4], p.190).

Remark 3. Peka Koskela ([5], 5.10) verified that SC domains are weakly linearly locally connected and hence locally connected at each finite boundary point ([4], p.190).

Remark 4. Weak linearly locally connected domains are finitely connected at infinity ([4], corollary 2.3) and hence finitelly connected at each boundary point.

Theorem 6. *Suppose* $D \subset \overline{R}^n$ *is QED (or* $D \subset R^n$ *is SC). Then* D *has property* P_1 *at each (finite) boundary point ([4], Lemma 3.1).*

Theorem 7. *Suppose* $D \subset \overline{R}^n$ *is QED (or* $D \subset R^n$ *is SC). Then* D *has property* P_2^r *at each (finite) boundary point.*

Proof. We assume that $D \subset R^n$ is SC and let $b \in \partial D$ be a finite point. Let b_1 be a boundary point of D, $b \neq b_1$. We will assume that $b_1 \neq \infty$. If $b_1 = \infty$ then we consider the coresponding spherical distance ([7], Definition 12.1).

Let $r = |b_1 - b|$ and let ε and r_1 be such that $0 < \varepsilon < r_1 < r$. Let E be a connected set in D, such that $b, b_1 \in \overline{E}$. There exists a closed arc $\gamma \subset \overline{E} \cap \overline{B}(b, r)$ with end points b and b_1. We consider $\gamma_1 = \gamma \cap \overline{B}(b, r_1)$. Choose a continuum $K \subset \overline{D \cap \overline{B}(b, r_1)}$ joining the spheres $S(b, \varepsilon)$, $S(b, r_1)$ and $K \cap E = \Phi$. Since $K, \gamma_1 \subset \overline{D \cap B(b, r)}$ are continua and $2r > \min\{diam(K), diam(\gamma_1)\} = d$, by 2.7 [4] it follows that there exists a constant c which depends only on the parameters r, d, n such that

$$s - cap(\gamma_1, K; D) \leq c + 2^n \cdot cap(\gamma_1, K; D) \tag{1}$$

Since each sphere $S(b, t)$ meets both sets γ_1, K for $\varepsilon < t < r_1$, according to Theorem 10.12 [7], we obtain

$$c_n \log \frac{r_1}{\varepsilon} \leq M(\Delta(\gamma_1, K)) \tag{2}$$

where $c_n > 0$ is as defined in [7] (10.11).

On the other hand, by Theorem 1 [1],

$$cap(\gamma_1, K; D) = M(\Delta(\gamma_1, K; D)). \tag{3}$$

By combining (1), (2), (3) and since D is SC,

$$c_n \log \frac{r_1}{\varepsilon} \leq M(\Delta(\gamma_1, K)) \leq s - cap(\gamma_1, K) \leq M \cdot s - cap(\gamma_1, K; D) \leq$$
$$\leq M \cdot [c + 2^n \cdot M(\Delta(\gamma_1, K; D))] \leq M \cdot [c + 2^n \cdot M(\Delta(E, K; D))].$$

It follows that
$$M(\Delta(E,K;D)) \geq \frac{c_n \log\frac{r_1}{\varepsilon} - Mc}{M \cdot 2^n} = \delta$$
and hence D has property P_2^* at b.

If D is QED, with same notations we have
$$c_n \log\frac{r_1}{\varepsilon} \leq M(\Delta(\gamma_1, K)) \leq k \cdot M(\Delta(\gamma_1, K; D)) \leq k \cdot M(\Delta(E, K; D))$$
and hence
$$M(\Delta(E,K;D)) \geq \frac{c_n}{k} \log\frac{r_1}{\varepsilon} = \delta'$$
so D has property P_2^* at b.

Theorem 8. *Suppose that $f : D \to D'$ is a quasiconformal mapping and $D' \subset \overline{R}^n$ is QED (or $D' \subset R^n$ is SC). Let $z \in \partial D$ and $w \in C(f,z), (\infty \notin C(f,z))$. If D has properties P_1 at each point of $C(f^{-1},w)$ and property P_2^* at least at a point of $C(f^{-1},w)$, then f can be extended to a homeomorphism $f^* : D \cup \{z\} \to D' \cup \{w\}$.*

Proof. Since D has property P_1 at every point of $C(f^{-1},w)$, D has property P_1 at z, by Remarks 2, 3, 4, D' is finitely connected on boundary. Applying Corollary 17.14 [7], we conclude that $C(f,z)$ has a single point, i. e. $C(f,z) = \{w\}$. On the other hand D has property P_2^* at least at a point of $C(f^{-1},w)$, $f^{-1} : D' \to D$ is a quasiconformal mapping and D' is locally connected at w. By Theorem 17.15 [7] with the remark that it is true if instead of property P_2 one considers property P_2^*, we have $C(f^{-1},w) = \{z\}$. Therefore the theorem is proved.

Corollary 1. *Suppose that $f : D \to D'$ is a quasiconformal mapping and $D' \subset \overline{R}^n$ is QED (or $D' \subset R^n$ is SC, $\infty \notin \partial D'$). Then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}'$ if and only if D has properties P_1 and P_2^* on the boundary.*

Proof. The sufficiency follows by Theorem 8. We prove the necessity. Let z be a boundary point of D and $f^*(z) = w$. By theorems 6, 7, D' has properties P_1 and P_2^* at w. Since $(f^*)^{-1} : \overline{D}' \to \overline{D}$ is a homeomorphism, using Remark 17.8 [7] it follows that D has properties P_1 at z. We prove that D has property P_2^* at z. In order to facilitate the writing, we denote $(f^*)^{-1} = g$. Let z_1 be a boundary point of D, $z_1 \neq z$. Since g is a homeomorphism, there exists $w_1 \in \partial D', w_1 \neq w$ such that $g(w_1) = z_1$. By the fact that D' has property P_2^* at w, it follows that there exists a continuum $F' \subset D'$ and $\delta > 0$ such that
$$M(\Delta(E', F'; D')) \geq \delta, \tag{1}$$
whenever E' is connected subset of D' with $w, w_1 \in \overline{E}'$.

Now set $F = g(F')$ and $\delta' = \frac{\delta}{K(g)}$. We will show that the condition from Definition 3 is satisfied by this F and by δ'. Using (1) and the fact that $g/_{D'}$ is quasiconformal, we obtain
$$M(\Delta(E,F;D)) \geq \frac{M(\Delta(E', F'; D'))}{K(g)} \geq \frac{\delta}{K(g)} = \delta',$$
whenever E is connected subset of D with $z, z_1 \in \overline{E}$.

Hence, D has property P_2^* at z.

Theorem 9. *Let $f : D \to D'$ be a quasiconformal mapping. If $D \subset \overline{R}^n$ is QED (or*

$D \subset R^n$ is SC, $\infty \notin \partial D$) then f can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D}'$ if and only if D' is locally connected on the boundary ([4], Corollary 3.5 (c)).

Proof. The proof is based on the Theorems 6, 7. First we show the necessity. By Remarks 2, 3, D is locally connected on the boundary, and by the Theorem 3.1 [6] we have that D' is locally connected on the boundary.

For the sufficiency, suppose that D' is locally connected on the boundary. Let b be a boundary point of D. By Theorem 6, D has property P_1 at b and use Corollary 17.14 [7] we obtain that there exists $\lim\limits_{x \to b} f(x)$. Let b' be a boundary point of D' and hence D' is locally connected at b'. By Theorem 7, D has property P_2^* on the boundary and in accordance with Theorem 17.15 [7] and the remark that it is true if instead of property P_2 one considers property P_2^*, there exists $\lim\limits_{y \to b'} f^{-1}(y)$ and the proof is complete.

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