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On Morley's Miracle Theorem

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Abstract. We use work by A. Connes to explore possible generalizations of Morley's trisector theorem to triangles in arbitrary valued fields. We find that Morley's theorem is essentially an unique phenomenon. However, tri-sectioning procedures different from Morley's do exist for generic sets of triangles.

Résumé (Sur le Théorème Miracle de Money). Nous utilisons le travail dévélopé par A. Connes afin d'explorer des généralisations possibles du théorème trisectrice de Morley dans des corps valués arbitraires. Nous d´ecouvrons que le théorème de Morley représent un phénomène essentiellement unique. Toutefois, des procédures de trisectionnement différentes de celle de Morley existent pour des ensembles génériques de triangles.

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Soit $(K, |\cdot|)$ un corps commutatif valué et soit $T = (v_1, v_2, v_3)$ un triangle dans K. Nous inspirant de Connes [2], nous considérons des "trisectrices" arbitraires g_i, g_i' (c.à.d., des rotations de K autour du sommet v_i), $i = 1, 2, 3$ telles que les trisectrices "adjacentes" g_i' et g_{i+1} viennent "s'intersecter" à w_i (c.à.d., $g_i' \circ g_{i+1}$ a un seul point fixe, w_i) et telles que (w_1, w_2, w_3) est un triangle équilatéral. Nous nous proposons de trouver touts les corps $(K, |\cdot|)$ qui possèdent des fonctions globales, $g_i(T)$, $g_i'(T)$, $i = 1, 2, 3$, qui dependent effectivement seulement de l'angle $\theta_i(T)$ du triangle T au sommet v_i. Nous appelons ceci le problème de Morley parce que c'est à lui que l'on doit la fameuse découverte de 1899 montrant que le corps complex $\mathbf{C}$, équippé avec la valeur absolue standard $|\cdot|_0$, est un tel corps, avec trisectionnement donné à parts égales (le théorème trisectrice de Morley).

Nous dècouvrons que le théorème de Morley est un phénomène rigide. C'est à dire:

a) Le problème de Morley a une solution dans $(K, |\cdot|)$ seulement si $(K, |\cdot|)$ est un sous-corps de $(\mathbf{C}, |\cdot|_0)$.

b) La solution trisectrice de Morley est l'unique solution du problème de Morley pour un sous-corps de $(\mathbf{C}, |\cdot|_0)$, si deux conditions naturelles en rapport avec l'internalité et la continuité des trisectionnements sont satisfaites. Particulièrment, le sous-corps doit contenir les racines cubiques complexes de toutes ses éléments de valeur absolue 1.

c) Il existe des solutions du problème de Morley qui ne sont pas des solutions trisectrices de Morley, pour des ensembles génériques des triangles.

1 Introduction

Around 1899 F. Morley proved a remarkable theorem on the elementary geometry of Euclidean triangles: In any triangle, the intersections of adjacent trisectors form the vertices of an equilateral triangle. For a detailed account of the fascinating history of this problem the reader is directed to A. Bogomolny's Web page [1].

A hundred years after that A. Connes has given a beautiful proof to Morley's theorem by considering the group of affine transformations of the line over an arbitrary field [2]. In this Note we announce that Connes' work can be generalized in two directions: first, by doubling the number of affine transformations, the concept of trisector can be replaced by arbitrary angle tri-sectioning, and second, the classical Morley set-up, which corresponds to the Euclidean plane (identified with the field of complex numbers) can be extended to arbitrary valued fields.

Our main finding (see Theorems 3.2 and 3.5 below) is that Morley's theorem is an unique phenomenon, in the following sense:

a) A valued field for which Morley's problem has a solution must necessarily be (isomorphic to) a subfield of the field of complex numbers, with absolute value equivalent to the standard one.

b) For a subfield of the field of complex numbers no other (continuously varying with the angle) procedure of internally tri-sectioning the angles of a triangle, except Morley's, yields equilateral triangles. Therefore, the subfield must contain the cube roots of all its elements of absolute value one.

We also find (Theorem 4.1), somewhat surprisingly, that for a generic set of complex triangles there are solutions to Morley's problem totally different from Morley's.

The proofs of a number of very technical results will only be sketched here. Full details will appear elsewhere.

2 A General Result

Let K be a *valued field* of arbitrary characteristic equipped with an absolute value function $|\cdot| : K \to \mathbb{R}_+$. Typical examples of valued fields will be the familiar fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, equipped with the standard absolute value $|\cdot|_0$, or $\mathbb{Q}$, equipped with the p-adic absolute value $|\cdot|_p$, p a prime number.

An ordered triple (v_1, v_2, v_3) of points in K is said to form a *triangle* if the three possible triangle inequalities hold, e.g., $|v_1 - v_2| < |v_1 - v_3| + |v_3 - v_2|$, etc. A triangle will be *equilateral* if $|v_1 - v_2| = |v_2 - v_3| = |v_3 - v_1|$. Equilateral triangles admit a simple characterization.

Proposition 2.1. *Let* v_1, v_2, v_3, *be points in* K. *Then* (v_1, v_2, v_3) *is an equilateral triangle if and only if* $v_1 \neq v_2$ *and*

$$\frac{v_3 - v_1}{v_1 - v_2} \in \Delta, \ \textit{where} \ \ \Delta = \{j \in K \mid |j| = |j+1| = 1\}.$$

We note that $j \in \Delta$ if and only if $(0, 1, -j)$ is an equilateral triangle. Δ is empty if $K = (\mathbb{Q}, |\cdot|_0)$, or $K = (\mathbb{R}, |\cdot|_0)$, and if $K = (\mathbb{C}, |\cdot|_0)$ it equals the set of nontrivial cube roots of unity, i.e., $\Delta = \{j \in \mathbb{C} \mid j^2 + j + 1 = 0\}$. When $K = (\mathbb{Q}, |\cdot|_p)$, Δ is infinite.

Let $\mathcal{G}$ denote the group of *affine transformations* of K, that is

$$\mathcal{G} = \{g : K \to K \mid g(x) = ax + b, a \in K \setminus \{0\}, b \in K\}$$

and let $\mathcal{T} = \{g \in \mathcal{G} \mid a = 1\}$ be its subgroup of *translations*. Clearly, $\mathcal{G}$ acts on the set of all triangles, leaving invariant the equilateral ones. Any element in $g \in \mathcal{G} \setminus \mathcal{T}$ admits one and only one fixed point, namely $\text{fix}(g) = b/(1-a)$. Let also $\mathcal{R} = \{g \in \mathcal{G} \mid |a| = 1, a \neq 1\}$ be the set of *proper rotations* of K (about their fixed points).

Consider now a triangle (v_1, v_2, v_3) in K and six elements, g_i, g_i', all in $\mathcal{R}$, $\text{fix}(g_i) = \text{fix}(g_i') = v_i$, $i = 1, 2, 3$. g_i and g_i' will be interpreted as arbitrary "*trisectors*" of the "*angle at* v_i" of the triangle (v_1, v_2, v_3). Assume that the *adjacent* trisectors g_i' and g_{i+1}, *intersect*, in the sense that $g_i' \circ g_{i+1} \notin \mathcal{T}$. (Obviously, here we permute the index i circularly, so $i + 1 = 1$ when $i = 3$). Specifically, we have, for $i = 1, 2, 3$,

$$g_i(x) = a_i x + b_i, \quad g_i'(x) = a_i' x + b_i', \quad a_i, a_i', b_i, b_i' \in K, \quad |a_i| = |a_i'| = 1, \quad a_i, a_i' \neq 1.$$

Then the restrictions imposed above amount to

$$\frac{b_i}{1 - a_i} = \frac{b_i'}{1 - a_i'} = v_i, \quad a_i' a_{i+1} \neq 1, \quad i = 1, 2, 3.$$

Let us denote by w_i the fixed point of $g_i' \circ g_{i+1}$, i.e., the intersection point of the adjacent trisectors g_i' and g_{i+1}. Then

$$w_i = \frac{(1 - a_i')v_i + a_i'(1 - a_{i+1})v_{i+1}}{1 - a_i' a_{i+1}}, \quad i = 1, 2, 3. \tag{2.2}$$

The goal is to find suitable conditions on a_i, a_i', under which (w_1, w_2, w_3) becomes an equilateral triangle. These conditions should depend only on the "angles" of the triangle (v_1, v_2, v_3) and not the triangle itself. It is therefore necessary to define a suitable notion of "angles of a triangle" in a valued field.

Definition 2.3. *Given a triangle* (v_1, v_2, v_3) *in a valued field* $(K, |\cdot|)$, *an ordered triple* $(\theta_1, \theta_2, \theta_3)$ *of elements of* K, $|\theta_i| = 1$, $\theta_i \neq 1$, $i = 1, 2, 3$, *is called a choice of angles for that triangle if the proper rotations* $h_i(x) = \theta_i x + v_i(1 - \theta_i)$, $i = 1, 2, 3$, *satisfy*

$$h_1 \circ h_2 \circ h_3 \ = \ 1_{\mathcal{G}} \tag{2.4}$$

For the choice of angles $(\theta_1, \theta_2, \theta_3)$, θ_i will then be called the angle of the triangle at the vertex v_i.

A routine calculation shows that Equation (2.4) is equivalent to

$$\theta_1\theta_2\theta_3 = 1 \quad \text{and} \quad \frac{\theta_1(1-\theta_2)}{1-\theta_1\theta_2} = \frac{v_3 - v_1}{v_2 - v_1}. \tag{2.5}$$

It is clear from (2.5) that two triangles whose vertices correspond under an affine transformation have equal angles. A triangle may not have angles at all ($K = \mathbb{Z}_3$), or have several choices of angles $(\mathbb{Q}, |\cdot|_p)$, but what really justifies Definition 2.3 is what happens in $(\mathbb{C}, |\cdot|_0)$.

Proposition 2.6. *In $(\mathbb{C}, |\cdot|_0)$ each triangle (v_1, v_2, v_3) has an unique choice of angles $(\theta_1, \theta_2, \theta_3)$, given by*

$$\theta_i = \frac{v_{i+2} - v_i}{\overline{v_{i+2}} - \overline{v_i}} \frac{\overline{v_{i+1}} - \overline{v_i}}{v_{i+1} - v_i}, \quad i = 1, 2, 3, \tag{2.7}$$

where the bar *operation denotes complex conjugation. θ_i is related to the measure α_i, $-\pi < \alpha_i < \pi$, of the oriented Euclidean angle made by the Euclidean rays $\overrightarrow{v_iv_{i+1}}$ and $\overrightarrow{v_iv_{i+2}}$ by $\theta_i = \exp\left(2\sqrt{-1}\alpha_i\right)$.*

Here comes the central result of this section.

Theorem 2.8. *Let (v_1, v_2, v_3) be a triangle in a valued field $(K, |\cdot|)$ and assume that $(\theta_1, \theta_2, \theta_3)$ is a choice of angles for it. Let $a_i, a_i' \in K$ be such that $a_i, a_i' \neq 1$, $a_i'a_{i+1} \neq 1$, $|a_i| = |a_i'| = 1$, $i = 1, 2, 3$. Let*

$$w_i = \frac{(1-a_i')v_i + a_i'(1-a_{i+1})v_{i+1}}{1 - a_i'a_{i+1}}, \quad i = 1, 2, 3.$$

be the trisector intersection points given by Equation 2.2. Then (w_1, w_2, w_3) is an equilateral triangle if and only if the following condition holds, for some element $j \in \Delta$:

$$\begin{aligned}
&(1 - a_3'a_1)(-1 + a_1' + a_2' - a_1'a_2'a_2 - a_1'a_2'a_3 + a_1'a_2'a_2a_3 - a_2'\theta_1 + a_2'a_3\theta_1 + \\
&\quad a_1'a_2'a_2\theta_1 - a_1'a_2'a_2a_3\theta_1 + \theta_1\theta_2 - a_1'\theta_1\theta_2 - a_2'a_3\theta_1\theta_2 + a_1'a_2'a_3\theta_1\theta_2)j + \\
&(1 - a_2'a_3)(a_1' - a_1'a_2 - a_1'a_3'a_1 + a_1'a_3'a_1a_2 - \theta_1 + a_3'\theta_1 + a_1'a_2\theta_1 - a_1'a_3'a_2\theta_1 + \\
&\quad \theta_1\theta_2 - a_1'\theta_1\theta_2 - a_3'\theta_1\theta_2 + a_1'a_3'a_1\theta_1\theta_2 + a_1'a_3'a_2\theta_1\theta_2 - a_1'a_3'a_1a_2\theta_1\theta_2) = 0.
\end{aligned} \tag{2.9}$$

$$a_1'(1 - a_3'a_1)(1 - a_2)(1 - \theta_1\theta_2) - (1 - a_1'a_2)(1 - a_3')\theta_1(1 - \theta_2) \neq 0. \tag{2.10}$$

Proof. (w_1, w_2, w_3) is an equilateral triangle if and only if $(g(w_1), g(w_2), g(w_3))$, $g \in \mathcal{G}$, is an equilateral triangle. For $g(x) = \frac{x - v_1}{v_2 - v_1}$, $(g(w_1), g(w_2), g(w_3)) = (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)$, where

$$\begin{aligned}
\widetilde{w}_1 &= \frac{a_1'(1-a_2)}{1 - a_1'a_2}, \quad \widetilde{w}_3 = \frac{(1-a_3')\theta_1(1-\theta_2)}{(1 - a_3'a_1)(1-\theta_1\theta_2)}, \\
\widetilde{w}_2 &= \frac{(1-a_2')(1-\theta_1\theta_2) + a_2'(1-a_3)\theta_1(1-\theta_2)}{(1-a_2'a_3)(1-\theta_1\theta_2)}.
\end{aligned} \tag{2.11}$$

Notice now that Equations (2.9) and (2.10) are a convenient implementation of Proposition 2.1 for the triangle $(\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)$. □

3 Applications

In this section we will present applications to Theorem 2.8.

Morley's theorem. *Let* $(K, |\cdot|)$ *be a valued field such that every element of* $\{a \in K \mid |a| = 1\}$ *admits three distinct cube roots in* K. *Let* (v_1, v_2, v_3) *be a triangle in* K *with a choice of angles* $(\theta_1, \theta_2, \theta_3)$. *Let* t_i *be a cube root of* θ_i, $i = 1, 2, 3$, *with the property that* $t_1t_2t_3 \neq 1$. *Consider the trisectors* $g_i = g_i'$, $i = 1, 2, 3$, *given by* $g_i(x) = g_i'(x) = t_ix + v_i(1 - t_i)$. *Then the intersections* w_i *of the adjacent trisectors* g_i' *and* g_{i+1}, $i = 1, 2, 3$, *form the vertices of an equilateral triangle.*

Proof. By hypothesis, $a_i = a_i' = t_i$ and $\theta_i = t_i^3$, $i = 1, 2, 3$. It is obvious that $a_i, a_i' \neq 1$, $a_i'a_{i+1} \neq 1$, $|a_i| = |a_i'| = 1$, $i = 1, 2, 3$. The proof consists in showing that this assignment of trisector angles satisfies Equations (2.9) and (2.10) of Theorem 2.8, if the element $j \in \Delta$ is taken to be $j = \dfrac{1}{t_1t_2t_3}$. □

The above theorem brings up a very natural problem.

Morley's problem. Given a valued field $(K, |\cdot|)$ find functions $a_i(T)$, $a_i'(T)$, $i = 1, 2, 3$, defined on the set of all triangles $T = (v_1(T), v_2(T), v_3(T))$ of K, with values in $\{a \in K \mid |a| = 1, a \neq 1\}$, which satisfy the following two conditions:

a) For each $i = 1, 2, 3$, $a_i'(T)a_{i+1}(T) \neq 1$, and the adjacent trisector intersection points $w_1(T)$, $w_2(T)$, $w_3(T)$, given by Equation 2.2, form the vertices of an equilateral triangle.

b) For each $i = 1, 2, 3$, $a_i(T), a_i'(T)$ depend only on the "i^{th} angle of the triangle T", in the sense that if T_1 and T_2 are two triangles in K and $\Theta_1 = (\theta_1^1, \theta_2^1, \theta_3^1)$, respectively $\Theta_2 = (\theta_1^2, \theta_2^2, \theta_3^2)$, are choices of angles for T_1, respectively T_2, such that $\theta_i^1 = \theta_i^2$, then $a_i(T_1) = a_i(T_2)$ and $a_i'(T_1) = a_i'(T_2)$.

Notice that Morley's problem makes sense regardless whether triangles have several choices of angles or none.

Theorem 3.1. *Morley's problem has no solution in a non-Archimedean field* $(K, |\cdot|)$ *with non-trivial absolute value.*

Proof. In non-Archimedean valued fields every three distinct points form a triangle and every triangle admits infinitely many choices of angles. As a result, any solution of the Morley's problem must necessarily consist in constant functions $a_i(T) = a_i$, $a_i'(T) = a_i'$, $i = 1, 2, 3$. However, this contradicts (2.9) and (2.10). □

Theorem 3.2. *If Morley's problem has a solution in a valued field* $(K, |\cdot|)$ *then* $(K, |\cdot|)$ *must be isomorphic to a subfield of the field of complex numbers, equipped with an absolute value equivalent to the standard one. In fact, it suffices to investigate the standard absolute value* $|\cdot|_0$.

Proof. By Theorem 3.1, $(K, |\cdot|)$ must be an Archimedean field. By Ostrovski's theorem [3], an Archimedean field is isomorphic (as valued field) with a subfield of $(\mathbb{C}, |\cdot|_0^q)$,

$0 < q \leq 1$. The triangles admitting angles are the same for both absolute values, $|\cdot|_0^q$, $0 < q \leq 1$, and $|\cdot|_0$, and therefore so is Morley's problem. □

We are left with analyzing the existence of solutions to Morley's problem for subfields K of $(\mathbb{C}, |\cdot|_0)$. We will assume that K contains the complex cube roots of unity since otherwise there are no equilateral triangles.

Proposition 3.3. *If $(K, |\cdot|_0)$ is a valued field as above, then any element of $\{a \in K \mid |a|_0 = 1,\ a \neq 1\}$, is an angle in some triangle. In other words, the set of angles is as large as possible. Moreover, this set of angles is dense in the unit circle $\mathbb{S}^1$ of $\mathbb{C}$.*

Proof. The proof of the first part of this proposition is a simple consequence of Proposition 2.6. The density claim follows from the fact that for any nontrivial cube root of unity j, $\mathbb{Q}(j) := \{a + bj \mid a, b \in \mathbb{Q}\}$ is a subset of K dense in $\mathbb{C}$. Then the angles $\frac{\bar{v}}{v}$, $v \in \mathbb{Q}(j)$, are dense in $\mathbb{S}^1$. □

The above proposition and Theorem 2.8 show that a solution to Morley's problem in $(K, |\cdot|_0)$ amounts to the existence of functions $a_i(\theta)$, $a_i'(\theta)$, $i = 1, 2, 3$, with domain and codomain $\{a \in K \mid |a|_0 = 1, a \neq 1\}$ such that if $(\theta_1, \theta_2, \theta_3)$ are angles in a triangle T then $a_i'(\theta_i)a_{i+1}(\theta_{i+1}) \neq 1$ and Equations (2.9) and (2.10) hold, if a_i, a_i', $i = 1, 2, 3$ are replaced by $a_i(\theta_i)$, $a_i'(\theta_i)$.

Definition 3.4. *Call a triangle T in $(K, |\cdot|_0)$ with angles*

$$(\theta_1, \theta_2, \theta_3) = \left(\exp\left(\sqrt{-1}\arg(\theta_1)\right), \exp\left(\sqrt{-1}\arg(\theta_2)\right), \exp\left(\sqrt{-1}\arg(\theta_3)\right)\right),$$

$0 < \arg(\theta_i) < 2\pi$, $i = 1, 2, 3$, positively oriented *if* $\sum_i \arg(\theta_i) = 2\pi$. *Otherwise, T will be* negatively oriented, *and* $\sum_i \arg(\theta_i) = 4\pi$.

We are now going to make two very reasonable assumptions, which guarantee that we are searching for solutions to Morley's problem which are most natural, in the sense that the tri-sectioning *a*) occurs always "inside the angles of triangles", and *b*) varies continuously with the angles of the triangle.

Assumption 1. We assume that the functions $a_i(\theta)$, $a_i'(\theta)$, $i = 1, 2, 3$, in a solution to Morley's problem, which are necessarily of type

$$f\left(\exp\left(\sqrt{-1}s\right)\right) = \exp\left(\sqrt{-1}\alpha(s)\right),\ 0 < s, \alpha(s) < 2\pi,$$

all satisfy the condition $\alpha(s) \leq s$ for every s.

Assumption 2. We also assume that a solution of Morley's problem in $(K, |\cdot|_0)$ has the property that the functions $a_i(\theta), a_i'(\theta), i = 1, 2, 3$ are *continuous* on $\{a \in K \mid |a| = 1, a \neq 1\}$ and that for positively, respectively negatively, oriented triangles j is the same in Equation 2.9.

We are now in a position to state the main result of this Note.

Theorem 3.5. *Let $(K, |\cdot|_0)$ be a subfield of the field of complex numbers equipped with the induced standard absolute value. Then Morley's problem admits a solution satisfying the above Assumptions 1 and 2 if and only if the complex cube roots of all the elements of $\{a \in K \mid |a|_0 = 1\}$ belong to K. Moreover, the solution is unique, namely Morley's trisector solution.*

Proof. The "if" part is essentially contained in Morley's Theorem, discussed at the beginning of this section. Indeed, if one defines the trisector functions a_i, a_i', $i = 1, 2, 3$, by the same formula

$$\exp(\sqrt{-1}s) \longmapsto \exp\left(\sqrt{-1}\frac{s}{3}\right), \quad 0 < s < 2\pi, \quad \exp(\sqrt{-1}s) \in K,$$

then these are clearly functions which satisfy the Assumptions 1 and 2 and all the hypotheses of Morley's Theorem.

The proof of the far more complicated "only if" part is a result of a very delicate, and lengthy, infinitesimal analysis of the situation at $\theta = 1$, and makes judicious use of the two assumptions. We sum up this infinitesimal analysis in the following two lemmas:

Lemma 3.6. a_i, a_i' $i = 1, 2, 3,$ *admit one-sided limits and derivatives as* $\theta \to 1$. *Moreover, all one-sided limits equal* 1 *and all one-sided derivatives equal* $1/3$, *as* $\theta \to 1, \Im(\theta) > 0$.

Lemma 3.7. $a_i^3(\theta) = (a_i'(\theta))^3 = \theta,$ $i = 1, 2, 3,$ *for* θ *near* 1, $\Im\theta > 0$.

End of proof of Theorem 3.5. a_i, a_i', $i = 1, 2, 3$, obey the conclusion of Lemma 3.7. Since by Assumption 1, $\arg a_i(\theta)) \leq \arg(\theta)$, $\arg a_i'(\theta)) \leq \arg(\theta)$, there exists $\rho > 0$, such that

$$a_i\left(\exp(\sqrt{-1}s)\right) = a_i'\left(\exp(\sqrt{-1}s)\right) = \exp\left(\sqrt{-1}\frac{s}{3}\right), \quad i = 1, 2, 3, \quad s \in (0, \rho). \tag{3.8}$$

In particular, (3.8) implies that the cube roots of all the angles $\theta \in K$, $\arg(\theta) \in (0, \rho)$, belong to K. Since every angle θ is the product of finitely many angles with small arguments, we conclude that the cube roots of all the elements in $\{a \in K| \ |a|_0 = 1\}$ belong to K. In order to finish the proof of the theorem we must show that the equations in (3.8) hold for $s \in (0, 2\pi)$. This can be proved by making use of Equation 2.9. □

4 An Example of a Near Solution

We end the Note with a (near) solution to Morley's problem in $(\mathbb{C}, |\cdot|_0)$ which is different from any of the 18 variants of Morley's cube root solution. The tri-sectioning functions a_i, a_i' will be continuous but the required restrictions $a_i'a_{i+1} \neq 1$ will fail on a small set of triangles.

Theorem 4.1. *For* $\theta \in \mathbb{S}^1 = \{a \in \mathbb{C} \mid |a|_0 = 1\}$ *define tri-sectioning functions* a_i, a_i',

$i = 1, 2, 3$, by the formulas,

$$\begin{aligned} a_1(\theta) = a_3'(\theta) &= \frac{1 + 3\theta + 2j\theta}{3 + j + \theta + j\theta}, \\ a_2(\theta) = a_2'(\theta) &= \theta, \\ a_3(\theta) = a_1'(\theta) &= \frac{\theta(7 + 5j - \theta + j\theta)}{1 - j + 5\theta + 7j\theta}, \end{aligned} \tag{4.2}$$

where $j = \exp\left(\sqrt{-1}\frac{2\pi}{3}\right)$ is one of the two non-trivial cube roots of unity in $\mathbb{C}$. Then the assignment (4.2) represents a solution to Morley's problem which works for all triangles (v_1, v_2, v_3) in $\mathbb{C}$, except those similar to triangles $(0, 1, v)$, v on the circle whose equation in polar coordinates, $z = r\exp\left(\sqrt{-1}\alpha\right)$, is $r = \sqrt{3}/3 \sin\alpha$. One can argue that for such triangles the corresponding equilateral triangles have vertices at infinity.

Proof. No lesser miracle that Morley's cube root solution itself, a_i, a_i' given by (4.2) satisfy Equation 2.9. In other words, for every set of angles $\left(\theta_1, \theta_2, \frac{1}{\theta_1\theta_2}\right)$ of some triangle, θ_1, θ_2, $a_1(\theta_1), a_1'(\theta_1)$, $a_2(\theta_2), a_2'(\theta_2)$, $a_3\left(\frac{1}{\theta_1\theta_2}\right), a_3'\left(\frac{1}{\theta_1\theta_2}\right)$, and $1/j$, satisfy (2.9). However, the constraints $a_i'a_{i+1} \neq 1$, $i = 1, 2, 3$, and (2.10) break down exactly for the triangles described in the theorem. □

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