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Remarks About the Drop Principle

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Abstract. A simplified version of the results in Danes [6] and Penot [10] about the equivalence between the drop principle (DP) and Ekeland's variational principle (EVP) is given. Further aspects involving the amended EVP are also discussed.

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1 Introduction

Let $(E, ||.||)$ be a Banach space. The mapping

(1D1) $\quad d(P,Q) = \inf\{||x-y||; x \in P, y \in Q\}, \quad P, Q \subseteq E$

is symmetric, non–transitive and non-sufficient; however, we call it the *distance* between P and Q. Moreover, its associated map (with infinite values)

(1D2) $\quad D(P,Q) = \sup\{||x-y||; x \in P, y \in Q\}, \quad P, Q \subseteq E$

is symmetric transitive and non-sufficient; this will be referred to as the *Distance* between P and Q. Denote also, for $C \subseteq E$, $\rho > 0$,

(1D3) $\quad E(C,\rho) = \{x \in E; d(x,C) < \rho\}, \quad E[C,\rho] = \{x \in E; d(x,C) \leq \rho\}$

(the *open/closed sphere* with center C and radius ρ). Note that, in view of

$$|d(x,C) - d(y,C)| \leq ||x-y||, \quad \text{for all} \quad x, y \in E, \tag{1.1}$$

this convention is acceptable from a topological viewpoint. Moreover, since

$$x \vdash d(x,C) \text{ is convex whenever } C \text{ is convex}, \tag{1.2}$$

these spheres are convex provided C is endowed with such a property. Further aspects are to be deduced from the metrical evaluation (1.1). For example, one has an evaluation like

$$d\,(P,Q) - (\rho+\sigma) \leq d(E[P,\rho], E[Q,\sigma]) \leq d(P,Q). \tag{1.3}$$

The proof is immediate; so, we do not give details.

Let us now describe the basic facts to be considered. For each point a in E and each subset C of E which does not contain it, denote

(1D4) $\quad \Delta(a, C) = \{\lambda a + (1-\lambda)c; \quad 0 \leq \lambda \leq 1, \quad c \in C\}.$

This will be referred to as the *drop* generated by a and C. In particular, when $C = \{c\}$, the drop in question is just

(1D5) $\quad \Delta(a, c) = [a, c]$ (the *segment* between a and c).

The following drop principle (in short: DP) is our starting point:

Theorem 1. *Let the parts A, C of E be such that*

(1H1) *A is closed (in the norm topology)*

(1H2) *C is closed, bounded and convex*

(1H3) *$d(A, C) \geq \rho$, for a certain $\rho > 0$.*

Then, for each point a_0 in A there exists some point a in A with

$$a \in \Delta(a_0, C) \quad (\text{hence } \; a \in A \cap \Delta(a_0, C)) \tag{1.4}$$

$$x \in A \cap \Delta(a, C) \Longrightarrow x = a \quad (\text{i.e., } \; A \cap \Delta(a, C) = \{a\}). \tag{1.5}$$

A first proof of Theorem 1, under the requirement

(1H4) $\quad C = E[c, \sigma], \quad$ for some $c \in E, \; \sigma > 0,$

was given in 1971 by Zabreiko and Krasnoselskii [16] (see also Danes [5]), through a metrical Cantor procedure. Further, in his 1974 paper, Brondsted [3] provided a different proof of the same, by using the Bishop-Phelps lemma [1]. This (cf. also Turinici [13]) shows that Theorem 1 is methodologically reductible to the 1974 Ekeland's variational principle [7] (in short: EVP) which may be stated as:

Theorem 2. *Let (M, d) be a complete metric space and $f : M \to]-\infty, \infty]$ be some function with*

(1H5) *f is lower semicontinuous on M*

(1H6) *$f \not\equiv \infty$ and $\inf[f(M)] > -\infty$.*

Then, for each a_0 in M with $f(a_0) < \infty$ there exists a in M with

$$d(a_0, a) \leq f(a_0) - f(a) \quad (\textit{hence } \; f(a) < \infty) \tag{1.6}$$

$$x \in M, \quad d(a, x) \leq f(a) - f(x) \Longrightarrow a = x. \tag{1.7}$$

The formal isomorphism between these statements raises the question of to what extent is retainable a converse implication like

$$\text{DP (the drop principle)} \Longrightarrow \text{EVP (Ekeland's variational principle)}. \tag{1.8}$$

A first answer to this was provided in 1985 by Danes [6] through the use of a certain geometric lemma. Further, in his 1986 paper, Penot [10] gave a second answer to the same, again by the use of a geometric lemma. It is our aim in the following to show that a certain simplification of the arguments above is still possible (so as to avoid the quoted lemmas); details are to be given in Section 4. This will be done by the use of a normed version of Ekeland's variational principle (to be found in Section 2). The converse to (1.8) question is being discussed in Section 3. And finally, Section 5 is devoted to the amended EVP, under the lines in Georgiev [8].

2 A normed version of EVP

It is our aim in the sequel to show that Ekeland's variational principle (Theorem 2) may be put into a normed form (useful to the converse question we just evoked). This may be phrased as

Theorem 3. *Let the (nonempty) part N of the Banach space $(V, \|.\|)$ and the function $g : V \to [-\infty, \infty[$ be such that (for some $\rho > 0$)*

(2H1) *N is (norm-)closed and* $\operatorname{diam}(N) \leq \rho$

(2H2) *g is upper semicontinuous on V*

(2H3) $0 = \inf[g(N)] \leq \sup[g(N)] < \rho$ *(hence $g \not\equiv -\infty$).*

Then, for each $b_0 \in N$ with $g(b_0) = 0$ there exists $b \in N$ with the properties

$$\|b - b_0\| \leq g(b) - g(b_0) \quad (= g(b)) \tag{2.1}$$

$$\|x - b\| > g(x) - g(b), \quad \textit{for all} \;\; x \in N, \; x \neq b. \tag{2.2}$$

That the implication below is true

$$\text{Theorem 2} \Longrightarrow \text{Theorem 3} \tag{2.3}$$

follows immediately by simply taking

$$(f : V \to]-\infty, \infty]) \qquad f(x) = \sup[g(N)] - g(x), \quad x \in V.$$

But the remarkable fact to be noted is that the converse relation also holds. This is shown from

Proposition 1. *We have the logical implication*

$$\textit{Theorem 3} \Longrightarrow \textit{Theorem 2.} \tag{2.4}$$

And so (cf. (2.3) above) these results are deductible from each other.

Proof. Let the complete metric space (M, d) and the function $f : M \to]-\infty, \infty]$ be as in (1H5)+(1H6). Further, let $a_0 \in M$ be such that $f(a_0) < \infty$; and take the number $\rho > 0$ in accordance with

(2H4) $(\inf[f(M)] \leq)\ f(a_0) < \inf[f(M)] + \rho/2.$

Without loss, one may now assume

(2H5) $\sup[f(M)] = f(a_0)$, as well as $\operatorname{diam}(M) \leq \rho$.

For, otherwise, the subset (of M) introduced as

(2D1) $\quad M_0 = \{x \in M;\ d(a_0, x) \leq f(a_0) - f(x)\}$

(which is closed (hence complete) in M) fulfils (2H5). And, from conclusions (1.6)+(1.7) written for M_0, it clearly follows the same fact about M. Denote

(2D2) $\quad V = \{\varphi : M \to R;\ \varphi$ is bounded and continuous$\}$.

This is a linear space which becomes a Banach one, with respect to

(2D3) $\quad ||\varphi|| = \sup\{|\varphi(x)|;\ x \in M\}, \quad \varphi \in V \quad$ (the *supremum* norm).

For each $z \in M$, let $T(z)$ stand for the element in V introduced as

(2D4) $\quad T(z)(x) = d(z, x) - d(a_0, x),\ \ x \in M.$

The canonical map $z \vdash T(z)$ fulfils

$$d(z, w) = ||Tz - Tw||, \quad z, w \in M; \tag{2.5}$$

so, it is an isometry between M and its image $N = T(M)$ in V. This, in particular, shows that (2H1) of Theorem 3 is true (with such a choice for N). Let the function $h : V \to]-\infty, \infty]$ be introduced as

(2D5) $\quad h(v) = f(T^{-1}(v)),\ \ v \in N \quad$ (hence $\ f(u) = h(T(u)),\ \ u \in M$).

It is not hard to verify (see for instance Danes [6]) that

$$h \text{ is lower semicontinuous on } V; \quad \text{and (cf. (2H5) (the first half))} \tag{2.6}$$

$$\sup[h(N)] = h(b_0), \quad \text{where } b_0 = T(a_0). \tag{2.7}$$

Denote further

(2D6) $\quad g(v) = h(b_0) - h(v), \quad v \in N;\ g(v) = -\infty,\ v \in V \setminus N.$

It is clear, via (2.6), that (2H2) is true (in this context). Moreover, (2.7) and (2H4) tell us that (2H3) is also true for this function. Summing up, Theorem 3 applies to our data. Hence, for the starting point $b_0 = T(a_0)$ in N, there exists $b = T(a)$ in N so that (2.1)+(2.2) be fulfilled. But, in this case, (1.6)+(1.7) are clear, by the very definition of g. The proof is thereby complete. ■

3 DP follows from EVP

Let the notations above be in use. From the remarks in Section 1, we know that the drop principle (Theorem 1) is reductible to the Ekeland's variational principle (Theorem 2). As a matter of fact, a stronger implication of this type is deductible. To describe it, the statement below (referred to as the strong EVP) is needed.

Theorem 4. *Let (M, d) be a complete metric space and $h : M \to [0, \infty[$, a function with*

(3H1) $\quad$ *h is continuous on M.*

Then, for each $a_0 \in M$ there exists $a \in M$ with the properties (1.6)+(1.7) (written for h).

Now, evidently,

$$\text{Theorem 2} \Longrightarrow \text{Theorem 4 (from a logical viewpoint).} \tag{3.1}$$

(The converse implication will be discussed in the next section). It is our main aim in the sequel to show that this variant of Theorem 2 will suffice for the objective to be attained.

Proposition 2. *We have the (logical) implication*

$$\textit{Theorem 4} \Longrightarrow \textit{Theorem 1.} \tag{3.2}$$

(Hence, the drop principle is deductible from Ekeland's).

Proof. Let the parts A, C of E and the number $\rho > 0$ be as in (1H1)–(1H3); and take some a_0 in A. Denote

(3D1) $\quad M = A \cap \Delta(a_0, C), \quad h(x) = \dfrac{\alpha}{\rho} d(x, C), \quad x \in M;$

where, by convention, $\alpha = D(M, C)$ (hence $\alpha \geq \rho$). It is not hard to verify (see, for instance, Turinici [13]) that

$$M \text{ is closed} \quad (\text{hence } (M, d) \text{ is complete}); \tag{3.3}$$

here, d is the metric induced by the norm. Let $x, y \in M$ be such that

(3H2) $\quad y \in \Delta(x, C)$ (hence, $y = \lambda x + (1 - \lambda)c$, for some $\lambda \in [0, 1]$, $c \in C$).

This gives on the one hand (via (1.2))

$$\left| \begin{array}{l} d(y, C) \leq \lambda d(x, C) \leq d(x, C); \quad \text{wherefrom} \\ (1 - \lambda)\rho \leq (1 - \lambda) d(x, C) \leq d(x, C) - d(y, C). \end{array} \right.$$

On the other hand (cf. the notations above)

$$||x - y|| = (1 - \lambda)\, ||x - c|| \leq (1 - \lambda)\alpha.$$

Combining these facts yields the property

$$x, y \in M,\ y \in \Delta(x, C) \Longrightarrow ||x - y|| \leq h(x) - h(y). \tag{3.4}$$

By Theorem 4 (applicable to (M, d) and h) there exists, for the starting point a_0 in M, some element a in M fulfiling (1.6)+(1.7) (written for h). We claim this is our desired point. Indeed, (1.4) is clear by the very definition of M. Further, let x be some element in $A \cap \Delta(a, C)$. Then,

$x \in A \cap \Delta(a_0, C) = M$ (because $a \in \Delta(a_0, C)$); $\quad$ and moreover

$d(a, x) \leq h(a) - h(x) \quad$ (if we take (3.4) into account).

But, in this case, (1.7) yields $x = a$; i.e., (1.5) holds as well. ∎

The following completion of this statement is useful for us. Denote

(3D2) $\quad C_\theta = E[C, \theta\rho], \quad \theta \geq 0.$

This is a family of closed bounded convex parts of E fulfiling (cf. Section 1)

$$C_0 = C \quad \text{and} \quad \theta \vdash C_\theta \quad \text{increases with } \theta \tag{3.5}$$

$$C_\mu \supseteq E[C_\nu, (\mu - \nu)\rho], \quad \text{whenever} \quad \mu \geq \nu \tag{3.6}$$

$$d(A, C_\theta) \geq (1-\theta)\rho, \quad \text{for each} \quad \theta \in [0,1]. \tag{3.7}$$

Theorem 5. *Let μ be arbitrary fixed in $]0,1[$. Then, for each a_0 in A there exists some point $a(\mu)$ in A with*

$$a(\mu) \in \Delta(a_0, C_\mu) \quad (\textit{hence} \quad a(\mu) \in A \cap \Delta(a_0, C_\mu)) \tag{3.8}$$

$$A \cap \Delta(a(\mu), C_\mu) = \{a(\mu)\}, \textit{ for each } v \textit{ in } [0, \mu]. \tag{3.9}$$

Moreover, for each ν in $]0,\mu[$, one has (with $\alpha(\nu) = D(a(\mu), C_\nu)$)

$$(\mu - \nu)\rho||x - a(\mu)|| \leq \alpha(\nu)d(x, A), \quad \textit{for all} \quad x \in \Delta(a(\mu), C_\nu). \tag{3.10}$$

Hence, in particular (with the same ν as before)

$$\textit{each} \quad (x_n) \subseteq \Delta(a(\mu), C_\nu) \quad \textit{with} \quad d(x_n, A) \to 0 \quad \textit{fulfils} \quad x_n \to a(\mu). \tag{3.11}$$

Proof. Let μ be taken as in the statement. By the admitted hypotheses, Theorem 1 is applicable to the triplet $(A, C_\mu, (1-\mu)\rho)$. So, for the given point a_0 in A there exists some element $a(\mu)$ in A with the properties (1.4)+(1.5) (written for C_μ). From this, (3.8)+(3.9) are clear. It remains now to prove (3.10). Fix ν in $]0,\mu[$ and let $x \neq a(\mu)$ be some point in $\Delta(a(\mu), C_\nu)$. Hence $x \notin A$ (by (3.9) plus $C_\nu \subseteq C_\mu$), as well as

(3H3) $\quad x = \theta a(\mu) + (1-\theta)c, \quad$ for some $\theta \in [0,1[, \ c \in C_\nu$.

Without loss, one may suppose that

(3H4) $\quad ||x - a(\mu)|| > d(x, A)$; $\quad$ for, otherwise, combining with

$$(\mu - \nu)\rho \leq (1-\nu)\rho \leq \alpha(\nu) \quad (\text{cf. } (3.7))$$

we are done. Let $\varepsilon > 0$ be taken so as $\varepsilon < ||x - a(\mu)|| - d(x, A)$. There exists, by definition, some point x_A of A with

$$||x - a(\mu)|| > d(x, A) + \varepsilon > ||x - x_A|| \ (> 0, \text{ in view of } \ x \notin A). \tag{3.12}$$

This, and (3.9), tells us that $x_A \notin \Delta(a(\mu), C_\mu)$. On the other hand, $x \in \text{int}\Delta(a(\mu), C_\mu)$; because $C_\nu \subseteq \text{int}C_\mu$ (cf. Turinici [14]). So, necessarily

$$[x_A, x] \cap \text{bd}\Delta(a(\mu), C_\mu) \neq \emptyset \quad (\text{where "bd" = the boundary}).$$

Let y be some point in this intersection. Note that $y \neq a(\mu)$; for, otherwise,

$$||x - a(\mu)|| \leq ||x - x_A|| < ||x - a(\mu)||,$$

contradiction.

Denote further (with $\theta \in [0,1[$ taken as in (3H3))

$$u = (y - \theta a(\mu))/(1-\theta) \quad (\text{hence } \; y = \theta a(\mu) + (1-\theta)u).$$

The case (=alternative) of

(3H5) $\quad d(u, C_\nu) < (\mu - \nu)\rho \quad$ (i.e., $u \in E(C_\nu, (\mu - \nu)\rho)$)

is not acceptable in our context. For (in view of (3.6)) $u \in \text{int}C_\mu \subseteq \text{int}\Delta(a(\mu), C_\mu)$; and this, combined with $a(\mu) \in \Delta(a(\mu), C_\mu)$, gives

$$y \in \text{int}\Delta(a(\mu), C_\mu) \text{ (see, for instance Bourbaki [2, ch.2, Sect.2]);}$$

in contradiction with the choice of y. Hence, $d(u, C_v) \geq (\mu - v)\rho$; wherefrom

$$||x - y|| = (1-\theta)\,||u - c|| \geq (1-\theta)(\mu - \nu)\rho. \tag{3.13}$$

On the other hand, we clearly have

$$||x - a(\mu)|| = (1-\theta)||a(\mu) - c|| \leq (1-\theta)\alpha(\nu);$$

so, by (3.12)+(3.13),

$$(\mu - \nu)\rho||x - a(\mu)|| \leq \alpha(\nu)||x - y|| \leq \alpha(\nu)||x - x_A|| \leq \alpha(\nu)(d(x, A) + \varepsilon).$$

As $\varepsilon > 0$ was arbitrarily chosen with respect to the precised condition, one gets (by a limit process) the desired evaluation (3.10). Hence the result. ■

Some remarks are in order. When A is bounded, conclusions of Theorem 1 are valid (cf. Phelps [11]) even if C were unbounded; precisely, when

(3H6) $\quad C$=(closed) convex cone of $\; E \;$ (in the usual sense).

This remark goes back to Bishop and Phelps [1]; see also Ursescu [15]. On the other hand, (1H3) may be substituted by

(3H7) $\quad d(A, C) = 0, \quad A \cap C = \emptyset, \qquad$ whenever

(3H8) $\quad E$ is reflexive and C = closed sphere in E (cf. (1H4);

we refer to Rolewicz [12] and Montesinos [9] for details. Finally, an exceptional set enlargement of these facts is possible, under the lines in Browder [4]; see Turinici [14] for a concrete result of this type.

4 A converse statement

We are now in position to discuss the converse to (3.2) implication (cf. Proposition 2). As already said in the introductory part, a positive answer to this is expectable, by the formal isomorphism between these statements. But, from a technical viewpoint, the situation is a bit more complicated, by the different nature of the concepts involved. This, among others, motivated the construction in Section 2 which may be viewed as an attempt of transposing the metrical concepts of Theorem 2 in a normed setting. The usefulness of it is to be judged from

Proposition 3. *We have the logical implication*

$$\text{Theorem 1} \Longrightarrow \text{Theorem 3}. \tag{4.1}$$

Hence (by Proposition 1) the variational Ekeland's principle is deductible from the drop principle.

Proof. Let the part N of the Banach space $(V, ||.||)$ and the function $g : V \to [-\infty, \infty[$ be such that conditions (2H1)-(2H3) of Theorem 3 are fulfilled (for some $\rho > 0$). Without loss, one may assume

(4H1) $\quad \gamma = \sup[g(N)] > 0 \quad$ (hence $0 < \gamma < \rho$).

Fix also $b_0 \in N$ with $g(b_0) = 0$ and take the numbers ω, λ according to

(4H2) $\quad 3 < \omega < 4 \quad$ (hence $\gamma + \rho < (\omega - 1)\rho$); $\quad \lambda = \dfrac{1}{\omega - 1}$.

Let us consider the product linear space $E = V \times R$, endowed with the maximum norm

(4D1) $\quad ||(x, \xi)|| = \max(\lambda||x||, |\xi|), \quad x \in V,\ \xi \in R.$

Clearly, $(E, ||.||)$ is a Banach space. Define further the subsets (of E)

(4D2) $\quad A = \{(x, \xi) \in N \times [0, \infty[; g(x) \geq \xi\}, \quad C = E[(b_0, \omega\rho), \rho].$

These fulfil conditions (1H1) and (1H2) respectively. Moreover, in view of

$$(y, \eta) \in C \Rightarrow |\eta - \omega\rho| \leq \rho \Rightarrow \gamma + \rho < (\omega - 1)\rho \leq \eta \leq (\omega + 1)\rho \tag{4.2}$$

one gets, for each $(x, \xi) \in A$, $(y, \eta) \in C$,

$$||(x, \xi) - (y, \eta)|| \geq |\xi - \eta| = \eta - \xi \geq (\omega - 2)\rho > \rho;$$

wherefrom

$$d(A, C) \geq \rho \quad \text{(i.e., (1H3) holds too)}. \tag{4.3}$$

Summing up, Theorem 1 applies to (A, C, ρ). It gives us, for the starting point $a_0 = (b_0, 0)$ in A, some point $a = (b, \beta)$ in A with the properties (1.4)+(1.5) above. We claim that $b \in N$ is our desired element. This may be shown under the lines below.

(i1) By (1.4), there must be some ν in $[0, 1]$ and some (y, η) in C with $b - b_0 = \nu(y - b_0)$, $\beta = \nu\eta$. The choice of C now gives

$$\lambda||y - b_0|| \leq \rho \quad (\text{hence } \lambda||b - b_0|| = \lambda\nu||y - b_0|| \leq \nu\rho \leq \rho);$$

and this shows that $(b, \omega\rho)$ also belongs to C. Moreover, the same fact yields, via (4.2) (and the choice of λ)

$$||b - b_0|| = \frac{\beta}{\eta}||y - b_0|| \leq \frac{\beta(\omega - 1)\rho}{(\omega - 1)\rho} = \beta. \tag{4.4}$$

Finally, by the definition of A, $\beta \le g(b) \le \gamma < \omega\rho$; wherefrom

$$(b, g(b)) = \mu(b,\beta) + (1-\mu)(b,\omega\rho), \quad \text{for some } \mu \in [0,1].$$

As a consequence of this, plus (1.5),

$$(b, g(b)) \in A \cap \Delta((b,\beta), C) = \{(b,\beta)\}.$$

Hence, necessarily, $g(b) = \beta$; and this gives (by means of (4.4))

$$\|b - b_0\| \le g(b) \ (= g(g) - g(b_0));$$

which is just conclusion (2.1).

(i2) Suppose by contradiction there would be some point z in N distinct from b, such that (2.2) be false; that is

(4H3) $\quad 0 < \|z - b\| \le g(z) - g(b) \ (= g(z) - \beta)$.

Put for simplicity $\zeta = \|z - b\| + \beta$; note that

$$\beta < \zeta \le g(z) \quad \text{(hence, in particular, } (z,\zeta) \in A). \tag{4.5}$$

We claim that there may be found some ε in $]0,1[$ with the properties

$$\lambda \left\| \frac{1}{\varepsilon}(z - b) + b - b_0 \right\| \le \rho, \quad \left| \frac{1}{\varepsilon}(\zeta - \beta) + \beta - \omega\rho \right| \le \rho. \tag{4.6}$$

To verify this, we note that the underlying relations are fulfiled as soon as

$$\frac{1}{\varepsilon}(\zeta - \beta) + \beta = (\omega - 1)\rho \quad \text{(see (4.4) above).}$$

(The assertion follows at once from the choice of (λ, ζ); we do not give details). This now forces us taking ε as the (unique) solution of the written equation; i.e., $\varepsilon = (\zeta - \beta) \,/\, [(\omega-1)\,\rho - \beta]$. That ε belongs to $]0,1[$ follows in essence (via (4H2) and (4.5)) from the developments above; precisely,

$$\left| \begin{array}{lcl} \beta = g(b) \le \gamma < (\omega - 1)\rho & \Longrightarrow & \varepsilon > 0 \\ \zeta \le g(z) \le \gamma < (\omega-1)\rho & \Longrightarrow & \varepsilon < 1. \end{array} \right.$$

Moreover, it fulfils, as above said, (4.6); and this proves our claim. But then, the point (y, η), where

$$y = \frac{1}{\varepsilon}(z - b) + b, \quad \eta = \frac{1}{\varepsilon}(\zeta - \beta) + \beta$$

belongs to C. And, from $z - b = \varepsilon(y - b)$, $\zeta - \beta = \varepsilon(\eta - \beta)$, one derives, via (4.5)

$$(z, \zeta) \in A \cap \Delta((b,\beta), C) = \{(b,\beta)\};$$

that is, $(z,\zeta) = (b,\beta)$, contradiction.
Hence, (4H3) is unacceptable; and relation (2.2) follows. Summing up, $b \in N$ fulfils conclusions of Theorem 3. Hence the result. ■

Now, by simply adding to this the developments of the preceding section, we get (cf. Danes [6] and Penot [10])

Proposition 4. *The drop principle (Theorem 1) is logically equivalent with the Ekeland's variational principle (Theorem 2). Hence, in particular, this last statement is deductible from its strong version (Theorem 4).*

5 The amended EVP

Let us now return to the framework of Section 1. The following local type result, referred to as the *amended* Ekeland variational principle (in short: amended EVP) may be stated.

Theorem 6. *Let (M, d) be a complete metric space and $f : M \to]-\infty, \infty]$ be a function satisfying (1H5)+(1H6). Then, for each $a_0 \in M$ with $f(a_0) < \infty$ and each $\varepsilon > 0$, there exist some $a = a(a_0, \varepsilon)$ in M and some $\lambda = \lambda(a_0, \varepsilon)$ in $]0, 1[$ with the properties*

$$d(a_0, a) \leq f(a_0) - f(a) + \varepsilon \tag{5.1}$$

$$\lambda d(a, x) < d(a, x) - f(a) + f(x), \quad \textit{for all } x \in M,\ x \neq a. \tag{5.2}$$

Hence, in particular, we have a conclusion like

$$\textit{each } (x_n) \subseteq M \quad \textit{with} \quad d(a, x_n) - f(a) + f(x_n) \to 0 \quad \textit{fulfils} \quad x_n \to a. \tag{5.3}$$

Proof. Let the number $\lambda = \lambda(a_0, \varepsilon)$ be taken in accordance with

(5H1) $\quad 0 < \lambda < 1$ and $\dfrac{\lambda}{1-\lambda}(f(a_0) - f_*) < \varepsilon$, where $f_* = \inf[f(M)]$.

Further, let the metric $(x, y) \vdash e(x, y)$ over M be introduced as

(5D1) $\quad e(x, y) = (1 - \lambda)d(x, y), \quad x, y \in M \quad$ (in short: $e = (1 - \lambda)d$).

By the admitted hypotheses, Theorem 2 is applicable to the complete metric space (M, e) and the same function f. Hence, for the starting point $a_0 \in M$ with $f(a_0) < \infty$, there exists $a \in M$ with the properties (1.6)+(1.7). The latter of these is just (5.2). And, by the former one, we derive (cf. (5H1))

$$\left| \begin{aligned} d(a_0, a) &\leq \frac{1}{1-\lambda}(f(a_0) - f(a)) = f(a_0) - f(a) + \frac{\lambda}{1-\lambda}(f(a_0) - f(a)) \\ &\leq f(a_0) - f(a) + \frac{\lambda}{1-\lambda}(f(a_0) - f_*) < f(a_0) - f(a) + \varepsilon; \end{aligned} \right.$$

wherefrom, (5.1) holds too. The proof is thereby complete. ■

In other words, the amended EVP is obtainable via "purely" metrical approaches. The geometric one, followed by Georgiev [8] is by using the statement in Section 3 to get first a normed version of this result. An interesting fact to be noted is that such a device is also working under the methods of the preceding section. Precisely, we have:

Theorem 7. *Let the (nonempty) part N of the Banach space $(V, ||.||)$ and the function $g: V \longrightarrow [-\infty, \infty[$ be as in (2H1)-(2H3) (for some $\rho > 0$). Then, for each $b_0 \in N$ with $g(b_0) = 0$ and each $\varepsilon > 0$ there exist $b = b(b_0, \varepsilon)$ in N and $\tau = \tau(b_0, \varepsilon)$ in $]0,1[$ such that*

$$||b - b_0|| \leq g(b) - g(b_0) + \varepsilon \ (= g(b) + \varepsilon) \tag{5.4}$$

$$\tau||x - b|| \leq g(b) + ||x - b|| - g(x), \quad \forall x \in N. \tag{5.5}$$

Hence, in particular, we have a conclusion like

$$\textit{each } \ (x_n) \subseteq N \ \textit{ with } \ g(b) + ||x_n - b|| - g(x_n) \to 0 \ \textit{ fulfils } \ x_n \to b. \tag{5.6}$$

Proof. Without loss, one may assume that (4H1) is true. Let the number ω be taken as in (4H2). We introduce the functions

$$\text{(5D2)} \qquad \varphi(t) = \frac{t}{\omega - t}, \ 0 < t < \omega; \quad \psi(t) = \frac{\varphi(1+t)}{\varphi(1)} - 1, \ 0 < t < \omega - 1.$$

Further, let us consider the cartesian product $E = V \times R$ endowed with the norm (4D1) (where $\lambda = \varphi(1)$). Finally, let the subsets A, C of E be taken as in (4D2); note that, with the notations of Section 3,

$$C_\theta = E[(b_0, \omega\rho), (1+\theta)\rho], \quad \text{for all } \ \theta \geq 0.$$

By the remarks in Proposition 3, conditions of Theorem 5 are holding for these data. So, given the starting point $a_0 = (b_0, 0)$ in A and μ in $]0, 1/4[$ (arbitrary for the moment) there exists $a(\mu) = (b, \beta)$ in A fulfiling (3.8)+(3.9) and endowed with the additional property: for each ν in $]0, \mu[$, the evaluation (3.10) is true. We now claim that b is our desired element (for a suitable choice of (μ, ν)). This will be shown in several steps.

(j1) By (4H2) and the preliminary choice of μ,

$$\omega - \frac{5}{2} > 2\mu \quad (\text{hence } \omega - 1 - 2\mu > \frac{3}{2}). \tag{5.7}$$

This, combined with the Lagrange mean value theorem, yields the evaluation (for all such μ)

$$\psi(\mu) \leq \frac{\mu\varphi'(1+\mu)}{\varphi(1)} = \frac{\omega\,(\omega-1)\,\mu}{(\omega - 1 - \mu)^2} \leq \frac{\omega\mu}{\omega - 1 - 2\mu} < \frac{2}{3}\omega\mu, \tag{5.8}$$

(j2) By (3.8), there must be some θ in $[0,1[$ and some (y, η) in C_μ with $b - b_0 = \theta(y - b_0)$, $\beta = \theta\eta$. The representation above for C_μ imposes to (y, η) a couple of conditions like

$$\left| \begin{array}{l} \varphi(1)||y - b_0|| \leq (1+\mu)\rho, \quad |\eta - \omega\rho| \leq (1+\mu)\rho \\ (\text{hence} \quad (\omega - 1 - \mu)\rho \leq \eta \leq (\omega + 1 + \mu)\rho) \end{array} \right.$$

This, combined with (5.8), yields

$$||b - b_0|| \leq \frac{\beta}{\eta}||y - b_0|| \leq \beta(1 + \psi(\mu)) < \beta + \frac{2}{3}\rho\omega\mu. \tag{5.9}$$

So, in order that (5.4) be true, we have to impose a condition like

$$\text{(5H2)} \qquad \frac{2}{3}\rho\omega\mu < \varepsilon \quad (\text{hence} \quad 0 < \mu < \min\left\{\frac{1}{4}, \frac{3\varepsilon}{2\rho\omega}\right\}).$$

(j3) It remains now to discuss (5.5). Let the pair (μ, ν) be taken in accordance with (5H2) and

$$\text{(5H3)} \qquad \mu < \omega - 3; \quad 2/3 < \nu/\mu < 1.$$

We show that, under these requirements, one has an evaluation like

$$(x, \beta + ||x - b||) \in \Delta((b, \beta), C_\nu), \quad \text{for all } \ x \in N. \tag{5.10}$$

In fact, take some x in N; without loss one may assume that $x \neq b$. The conclusion to be derived is clearly obtainable from

$$\left| \begin{array}{l} (x, \beta + ||x - b||) = (b, \beta) + \theta(y - b, \eta - \beta) \\ \text{for some } \theta \text{ in }]0, 1[\text{ and } (y, \eta) \text{ in } C_\nu. \end{array} \right.$$

This also reads (in an evident way)

$$(y = \frac{1}{\theta}(x - b) + b, \eta = \frac{1}{\theta}||x - b|| + \beta) \ \text{ is in } \ C_\nu;$$

or, equivalently

$$\varphi(1)\left\|\frac{1}{\theta}(x - b) + b - b_0\right\| \leq (1 + \nu)\rho, \quad \left|\frac{1}{\theta}\,||x - b|| + \beta - \omega\rho\right| \leq (1 + \nu)\rho. \tag{5.11}$$

A reasonable way of satisfying the second requirement above is by solving

$$\frac{1}{\theta}||x - b|| + \beta - \omega\rho = -(1 + \nu)\rho; \ \text{ hence } \ \theta = \frac{||x - b||}{(\omega - 1 - \nu)\rho - \beta}. \tag{5.12}$$

That $\theta > 0$ is clear, by the choice of ω. We also have $\theta < 1$; for, in view of (2H1) (the second half) and (5H3) (the first half),

$$||x - b|| \leq \rho < (\omega - 1 - \mu)\rho - \beta < (\omega - 1 - \nu)\rho - \beta.$$

This value of θ is also appropriate for the first half of (5.11). Indeed, a sufficient condition for the underlying relation is

$$\varphi(1)\left(\frac{1}{\theta}||x - b|| + ||b - b_0||\right) \leq (1 + \nu)\rho.$$

This (cf. (5.9) and (5.12)) is fulfilled as soon as

$$\varphi(1)[(\omega - 1 - \nu)\rho + \beta\psi(\mu)] \leq (1 + \nu)\rho; \ \text{ that is } \ \beta\psi(\mu) \leq \omega\rho\nu. \tag{5.13}$$

But, the obtained relation is clear, in view of (5.8) (and the choice (5H3) of (μ, ν)); hence (5.10) is retainable. Note that, as an immediate consequence of this and (3.9), one has

$$(x, \beta + ||x - b||) \notin A, \quad \text{for all } \ x \in N, \ x \neq b. \tag{5.14}$$

Summing up, conclusion (3.10) is applicable in the present context. To give it an appropriate form, note that (by the choice of λ)

$$\|(x, \beta + \|x - b\|) - (b, \beta)\| = \max(\lambda\|x - b\|, \|x - b\|) = \|x - b\|, \quad \forall x \in N.$$

On the other hand, by (5.14) above,

$$\left| \begin{array}{l} d((x, \beta + \|x - b\|), A) \leq \|(x, \beta + \|x - b\|) - (x, g(x))\| = \\ \beta + \|x - b\| - g(x) \leq g(b) + \|x - b\| - g(x), \quad \text{for all } x \in N. \end{array} \right.$$

This finally combined with

$$\alpha(\nu) \leq (\omega + 1 + \nu)\rho, \quad \text{for all such} \quad \nu$$

allows us writing the underlying conclusion under the convenient form

$$(\mu - \nu)\rho\|x - b\| \leq (\omega + 1 + \nu)\rho(g(b) + \|x - b\| - g(x)), \quad \forall\, x \in N.$$

But, the obtained relation is just (5.5), with $\tau = (\mu - \nu)/(\omega + 1 + \nu)$. The proof is thereby complete. ■

Now, by simply remembering the way of translating Theorem 2 in terms of Theorem 3, one re-obtains Theorem 6 (in this geometric way). Note that, the last conclusion (5.3) of this result was proved by Georgiev [8] through a different method, based on the statement in Phelps [11]. But the source relation (5.2) above seems to be new. Some interesting applications of the amended EVP to generic well-posed optimization problems were also provided in the quoted paper. But, a close analysis shows that all these may be also handled by means of the standard EVP. We shall develop such facts in a separate paper.

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