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On Some New Inequalities Similar to Certain Extensions of Hilbert Inequality

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Abstract. In this paper, we establish a new inequalities similar to certain extensions of Hilbert inequality.

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1 Introduction

In [1, p. 284] the following the extension of Hilbert inequality is given.

Theorem A. *If* $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} \geq 1$, $0 < \lambda = 2 - \frac{1}{p} - \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'} \leq 1$, *then*

$$\sum_{1}^{\infty} \sum_{1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} \leq k \left(\sum_{1}^{\infty} a_m^p \right)^{1/p} \left(\sum_{1}^{\infty} b_n^q \right)^{1/q}, \tag{1}$$

where $k = k(p, q)$ *depends on* p *and* q *only.*

The integral analogue of Theorem A can be stated as follows [1, p. 286].

Theorem B. *Under the same conditions as in Theorem A, we have*

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)\, g(y)}{(x+y)^{\lambda}} dx dy \leq k \left(\int_0^{\infty} p^p(x)\, dx \right)^{1/p} \left(\int_0^{\infty} g^q(y)\, dy \right)^{1/q} \tag{2}$$

where $k = k(p, q)$ *depends on* p *and* q *only.*

The inequalities in Theorem A and B were studied extensively and numerous variants, generalizations, and extensions appeared in the literature, see [2-7]. Recently, in [8] inequalities have given similar to the inequalities given in Theorems A and B. The main purpose of this paper is to establish some new inequalities similar to Theorem A and B, too. Our results provide new estimates on inequalities of this type.

2 Main results

Theorem 1. *Let $a(s)$ and $b(t)$ be real-valued nonnegative non-decreasing functions defined on N_m and N_n, respectively, where $N_m = \{0, 1, 2, \ldots, m\}$, $N_n = \{0, 1, 2, \ldots, n\}$ and define the operator ∇ by $\nabla u(t) = u(t) - u(t-1)$ for any non-decreasing function u defined on $N_0 = \{0, 1, 2, \ldots\}$. Let $p \geq 1$, $q \geq 1$ and $h > 1$, $\frac{1}{h} + \frac{1}{l} = 1$. Then*

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{hl\left(a(s)-a(0)\right)^{p}\left(b(t)-b(0)\right)^{q}}{l\cdot s^{k-1}+h\cdot t^{l-1}} \leq$$

$$\leq pq\cdot m^{(k-1)/k}\cdot n^{(l-1)/l}\left(\sum_{s=1}^{m}(m-s+1)\left(\nabla a(s)\cdot\left(a(s)-a(0)\right)^{p-1}\right)^{k}\right)^{1/k}\times$$

$$\times\left(\sum_{t=1}^{n}(n-t+1)\left(\nabla b(t)\cdot\left(b(t)-b(0)\right)^{q-1}\right)^{l}\right)^{1/l} \quad (3)$$

Proof. From the hypothesis, it is easy to observe that

$$a(s) - a(0) = \sum_{\tau=1}^{s}\nabla a(\tau), \quad s \in N_m \quad (4)$$

$$b(t) - b(0) = \sum_{\sigma=1}^{t}\nabla b(\sigma), \quad t \in N_n \quad (5)$$

By using the elementary inequality [1, p. 40], $x^p - y^q \leq px^{p-1}(x-y)$, where $x \geq 0$, $y \geq 0$ and $p \geq 1$, we have

$$\left(a(s+1)-a(0)\right)^{p} - \left(a(s)-a(0)\right)^{p} \leq p\left(a(s+1)-a(0)\right)^{p-1}\left(a(s+1)-s(s)\right) =$$

$$= p\left(a(s+1)-a(0)\right)^{p-1}\cdot\nabla a(s+1)$$

and

$$\sum_{s=0}^{k-1}\left(\left(a(s+1)-a(0)\right)^{p} - \left(a(s)-a(0)\right)^{p}\right) = \left(a(k)-a(0)\right)^{p} \leq$$

$$\leq p\sum_{s=0}^{k-1}\nabla a(s+1)\cdot\left(a(s+1)-a(0)\right)^{p-1} = \sum_{s=1}^{k}\nabla a(s)\cdot\left(a(s)-a(0)\right)^{p-1}$$

Thus

$$\left(a(s)-a(0)\right)^{p} \leq p\sum_{\tau=1}^{s}\nabla a(\tau)\cdot\left(a(\tau)-a(0)\right)^{p-1} \quad (6)$$

and similarly

$$\left(b(t)-b(0)\right)^{q} \leq q\sum_{\sigma=1}^{t}\nabla b(\sigma)\cdot\left(b(\sigma)-b(0)\right)^{q-1} \quad (7)$$

From (6), (7) and using Holder inequality and the elementary inequality [9].

$$xy \leq \frac{x^k}{h} + \frac{y^l}{l} \quad (8)$$

where $x \geq 0$, $y \geq 0$ and $\frac{1}{h} + \frac{1}{l} = 1$, $h > 1$ then

$$(a(s) - a(0))^p (b(t) - b(0))^q \leq pq \sum_{\tau=1}^{s} \nabla a(\tau) \cdot (a(\tau) - a(0))^{p-1} \times$$
$$\times \sum_{\sigma=1}^{t} \nabla b(\sigma) \cdot (b(\sigma) - b(0))^{q-1} \leq$$
$$\leq pq \cdot s^{(k-1)/k} \left(\sum_{\tau=1}^{s} \left(\nabla a(\tau) \cdot (a(\tau) - a(0))^{p-1} \right)^k \right)^{1/k} \times$$
$$\times t^{(l-1)/l} \left(\sum_{\sigma=1}^{t} \left(\nabla b(\sigma) \cdot (b(\sigma) - b(0))^{q-1} \right)^l \right)^{1/l} \leq$$
$$\leq \frac{pq \left(l \cdot s^{k-1} + h \cdot t^{l-1} \right)}{hl} \left(\sum_{\tau=1}^{s} \left(\nabla a(\tau) \cdot (a(\tau) - a(0))^{p-1} \right)^k \right)^{1/k} \times$$
$$\times \left(\sum_{\sigma=1}^{t} \left(\nabla b(\sigma) \cdot (b(\sigma) - b(0))^{q-1} \right)^l \right)^{1/l} \quad (9)$$

Dividing both sides of (9) by $\frac{l \cdot s^{k-1} + h \cdot t^{l-1}}{hl}$ and then taking the sum over t from 1 to n and then the sum over s from 1 to m and using Holder inequality, we observe that

$$\sum_{s=1}^{m} \sum_{t=1}^{n} \frac{hl (a(s) - a(0))^p (b(t) - b(0))^q}{l \cdot s^{k-1} + h \cdot t^{l-1}} \leq$$
$$\leq pq \sum_{s=1}^{m} \left(\sum_{\tau=1}^{s} \left(\nabla a(\tau) \cdot (a(\tau) - a(0))^{p-1} \right)^k \right)^{1/k} \times$$
$$\times \sum_{t=1}^{n} \left(\sum_{\sigma=1}^{t} \left(\nabla b(\sigma) \cdot (b(\sigma) - b(0))^{q-1} \right)^l \right)^{1/l} \leq$$
$$\leq pq \cdot m^{(k-1)/k} \left(\sum_{s=1}^{m} \sum_{\tau=1}^{s} \left(\nabla a(\tau) \cdot (a(\tau) - a(0))^{p-1} \right)^k \right)^{1/k} \times$$
$$\times n^{(l-1)/l} \sum_{t=1}^{n} \left(\sum_{\sigma=1}^{t} \left(\nabla b(\sigma) \cdot (b(\sigma) - b(0))^{q-1} \right)^l \right)^{1/l} =$$
$$= pq \cdot m^{(k-1)/k} \cdot n^{(l-1)/l} \left(\sum_{\tau=1}^{s} \left(\nabla a(\tau) \cdot (a(\tau) - a(0))^{p-1} \right)^k \sum_{s=\tau}^{m} 1 \right)^{k-1} \times$$
$$\times \left(\sum_{\sigma=1}^{t} \left(\nabla b(\sigma) \cdot (b(\sigma) - b(0))^{q-1} \right)^l \sum_{l=\sigma}^{n} 1 \right)^{l-1} =$$

$$= pq \cdot m^{(k-1)/k} \cdot n^{(l-1)/l} \left(\sum_{s=1}^{m} (m-s+1) \left(\nabla a(s) \cdot (a(s) - a(0))^{p-1} \right)^{k} \right)^{k-1} \times$$

$$\times \left(\sum_{t=1}^{n} (n-t+1) \left(\nabla b(t) \cdot (b(t) - b(0))^{q-1} \right)^{l} \right)^{l-1}.$$

Remark 1. We take $p = q = 1, a(0) = b(0) = 0$ in (3), the inequality (3) reduces to the following inequality

$$\sum_{s=1}^{m} \sum_{t=1}^{n} \frac{a(s)\, b(t)}{l \cdot s^{k-1} + h \cdot t^{l-1}} \leq \frac{m^{(k-1)/k} \cdot n^{(l-1)/l}}{hl} \left(\sum_{s=1}^{m} (m-s+1) (\nabla a(s))^{k} \right)^{1/k} \times$$

$$\times \left(\sum_{t=1}^{n} (n-t+1) (\nabla b(t))^{l} \right)^{1/l}. \quad (10)$$

This is just a new inequality similar to Theorem 1 which was given by B. G. Pachpatte in [8].

On the other hand, dividing both sides of (3) by $m^{(k-1)/k} \cdot n^{(l-1)/l}$ and then taking the sum over n from 1 to v and then the sum over m from 1 to u and using Holder inequality, we get following inequality

$$\sum_{m=1}^{u} \sum_{n=1}^{v} \left(\frac{m^{(k-1)/k}}{n^{(l-1)/l}} \sum_{s=1}^{m} \sum_{t=1}^{n} \frac{hl\, (a(s) - a(0))^{p} (b(t) - b(0))^{q}}{l \cdot s^{k-1} + h \cdot t^{l-1}} \right) \leq$$

$$\leq pq \cdot u^{(k-1)/k} \cdot v^{(l-1)/l} \left(\sum_{s=1}^{u} (u-s+1)(m-s+1) \left(\nabla a(s) \cdot (a(s) - a(0))^{p-1} \right)^{k} \right)^{1/k} \times$$

$$\times \left(\sum_{t=1}^{v} (v-t+1)(n-t+1) \left(\nabla b(t) \cdot (b(t) - b(0))^{q-1} \right)^{l} \right)^{1/l} \quad (11)$$

where u, v are two nature numbers.

Theorem 2. *Under the hypotheses of Theorem 1, if* $a(s) \neq b(0)$, $b(t) \neq b(0)$ *and* ϕ *and* ψ *are two real-valued nonnegative, submultiplicative, and monotone increasing convex functions defined on* R_{+}, *then*

$$\sum_{s=1}^{m} \sum_{t=1}^{n} \frac{hl\, (a(s) - a(0))\, (b(t) - b(0)) \cdot \phi^{p-1}(a(s) - a(0)) \cdot \psi^{q-1}(b(t) - b(0))}{l \cdot s^{k-1} + h \cdot t^{l-1}} \leq$$

$$\leq \phi(p)\, \psi(q) \cdot m^{(k-1)/k} \cdot n^{(l-1)/l} \left(\sum_{s=1}^{m} \left(\nabla a(s) \cdot \phi(a(s) - a(0))^{p-1} \right)^{k} (m-s+1) \right)^{1/k} \times$$

$$\times \left(\sum_{t=1}^{n} \left(\nabla b(t) \cdot \psi(b(t) - b(0))^{q-1} \right)^{l} (n-t+1) \right)^{1/l} \quad (12)$$

Proof. From (4) and inequality (6), we have

$$(a(s)-a(0))^{p-1} \le p\frac{\sum\limits_{\tau=1}^{s} \nabla a(\tau)\cdot(a(\tau)-a(0))^{p-1}}{\sum\limits_{\tau=1}^{s} \nabla a(\tau)}$$

From the hypotheses of Theorem 2 and using Jensen inequality and Holder inequality

$$\phi\left((a(s)-a(0))^{p-1}\right) \le \phi\left(p\frac{\sum\limits_{\tau=1}^{s} \nabla a(\tau)\cdot(a(\tau)-a(0))^{p-1}}{\sum\limits_{\tau=1}^{s} \nabla a(\tau)}\right) \le$$

$$\le \phi(p)\cdot\phi\left(\frac{\sum\limits_{\tau=1}^{s} \nabla a(\tau)\cdot(a(\tau)-a(0))^{p-1}}{\sum\limits_{\tau=1}^{s} \nabla a(\tau)}\right) \le$$

$$\le \frac{\phi(p)}{a(s)-a(0)}\sum_{\tau=1}^{s} \nabla a(\tau)\cdot\left(\phi\left(a(\tau)-a(0)\right)^{p-1}\right) \le$$

$$\le \frac{\phi(p)}{a(s)-a(0)}s^{(k-1)/k}\left(\sum_{\tau=1}^{s} \nabla a(\tau)\cdot\phi\left((a(\tau)-a(0))^{p-1}\right)^{k}\right)^{1/k} \tag{13}$$

and similarly

$$\psi\left((b(t)-b(0))^{q-1}\right) \le \frac{\psi(q)}{b(t)-b(0)}t^{(l-1)/l}\left(\sum_{\sigma=1}^{t}\left(\nabla b(\sigma)\cdot\psi\left((b(\sigma)-b(0))^{q-1}\right)\right)^{l}\right)^{l-1} \tag{14}$$

From (13), (14) and (8), we have

$$\frac{hl\,(a(s)-a(0))\,(b(t)-b(0))\,\phi\left((a(s)-a(0))^{p-1}\right)\psi\left((b(t)-b(0))^{q-1}\right)}{l\cdot s^{k-1}+h\cdot t^{l-1}} \le \phi(p)\,\psi(q)\cdot$$

$$\cdot\left(\sum_{\tau=1}^{s}\left(\nabla a(\tau)\cdot\phi\left((a(\tau)-a(0))^{p-1}\right)\right)^{k}\right)^{1/k}\left(\sum_{\sigma=1}^{t}\left(\nabla b(\sigma)\cdot\psi\left((b(\sigma)-b(0))^{q-1}\right)\right)^{l}\right)^{l-1} \tag{15}$$

Taking the sum on both sides of (15) first over t from 1 to n and then over s from 1 to m and using Holder inequality, we observe that:

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{hl\,(a(s)-a(0))\,(b(t)-b(0))\cdot\phi^{p-1}(a(s)-a(0))\cdot\psi^{q-1}(b(t)-b(0))}{l\cdot s^{k-1}+h\cdot t^{l-1}} \le$$

$$\leq \phi(p)\cdot\psi(q)\sum_{s=1}^{m}\left(\sum_{\tau=1}^{s}\left(\nabla a(\tau)\cdot\phi\left((a(\tau)-a(0))^{p-1}\right)\right)^{k}\right)^{1/k}\times$$

$$\times\sum_{t=1}^{n}\left(\sum_{\sigma=1}^{t}\left(\nabla b(\sigma)\cdot\psi\left((b(\sigma)-b(0))^{q-1}\right)\right)^{l}\right)^{l-1}\leq$$

$$\phi(p)\cdot\psi(q)\,m^{(k-1)/k}\cdot n^{(l-1)/l}\left(\sum_{s=1}^{m}\sum_{\tau=1}^{s}\left(\nabla a(\tau)\cdot\phi\left((a(\tau)-a(0))^{p-1}\right)\right)^{k}\right)^{1/k}\times$$

$$\times\left(\sum_{t=1}^{n}\sum_{\sigma=1}^{t}\left(\nabla b(\sigma)\cdot\psi\left((b(\sigma)-b(0))^{q-1}\right)\right)^{l}\right)^{l-1}=$$

$$=\phi(p)\cdot\psi(q)\,m^{(k-1)/k}\cdot n^{(l-1)/l}\left(\sum_{\tau=1}^{s}\left(\nabla a(\tau)\cdot\phi\left((a(\tau)-a(0))^{p-1}\right)\right)^{k}(m-\tau+1)\right)^{1/k}\times$$

$$\times\left(\sum_{\sigma=1}^{n}\left(\nabla b(\sigma)\cdot\psi\left((b(\sigma)-b(0))^{q-1}\right)\right)^{l}(n-\sigma+1)\right)^{1/l}=$$

$$=\phi(p)\cdot\psi(q)\,m^{(k-1)/k}\cdot n^{(l-1)/l}\left(\sum_{s=1}^{m}\left(\nabla a(\tau)\cdot\phi\left((a(\tau)-a(0))^{p-1}\right)\right)^{k}(m-s+1)\right)^{1/k}\times$$

$$\times\left(\sum_{t=1}^{n}\left(\nabla b(\sigma)\cdot\psi\left((b(\sigma)-b(0))^{q-1}\right)\right)^{l}(n-t+1)\right)^{1/l}.$$

Remark 2. It is obvious that the inequality (12) reduces to the inequality (10).

On the other hand, we can also get following inequality similar to (11).

$$\sum_{m=1}^{u}\sum_{n=1}^{v}\left(\frac{m^{(k-1)/k}}{n^{(l-1)/l}}\cdot\right.$$

$$\left.\cdot\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{hl\,(a(s)-a(0))\,(b(t)-b(0))\,\phi\left((a(s)-a(0))^{p-1}\right)\cdot\psi\left((b(t)-b(0))^{q-1}\right)}{l\cdot s^{k-1}+h\cdot t^{l-1}}\right)\leq$$

$$\leq\phi(p)\cdot\psi(q)\,u^{(k-1)/k}\cdot v^{(l-1)/l}\cdot$$

$$\cdot\left(\sum_{s=1}^{u}(u-s+1)\,(m-s+1)\left(\nabla a(s)\cdot\phi\left((a(s)-a(0))^{p-1}\right)\right)^{k}\right)^{1/k}\times$$

$$\times\left(\sum_{t=1}^{v}(v-t+1)\,(n-t+1)\left(\nabla b(t)\cdot\psi\left((b(t)-b(0))^{q-1}\right)\right)^{l}\right)^{1/l}\qquad(16)$$

3 Integral analogues

Theorem 3. *Let $f(s)$ and $g(t)$ be two real-valued nonnegative, non-decreasing continuous functions defined on $[0,x)$ and $[0,y)$, respectively, where x and y are positive real numbers. Let $p \geq 1$, $q \geq 1$ and $\frac{1}{h}+\frac{1}{l}=1$, $h>1$, then*

$$\int_0^x \int_0^y \frac{hl\left(f^p(s)-f^p(0)\right)\left(g^q(t)-g^q(0)\right)}{l \cdot s^{k-1}+h \cdot t^{l-1}} dsdt \leq pqx^{(k-1)/k} \cdot y^{(l-1)/l} \times$$

$$\times \left(\int_0^x (x-s)\left(f'(s) f^{p-1}(s)\right)^k ds\right)^{1/k} \left(\int_0^y (y-t)\left(g'(t) g^{q-1}(t)\right)^l dt\right)^{1/l} \quad (17)$$

Proof. From the hypotheses, we have

$$f^p(s)-f^p(0)=p \int_0^s f'(\tau) f^{p-1}(\tau) d\tau, \quad s \in [0,x), \quad (18)$$

$$g^q(t)-g^q(0)=q \int_0^t g'(\sigma) g^{q-1}(\sigma) d\sigma, \quad t \in [0,y) \quad (19)$$

From (18) and (19) and using Holder integral inequality and the elementary inequality (8), we have

$$\left(f^p(s)-f^p(0)\right)\left(g^q(t)-g^q(0)\right) \leq$$

$$\leq pq \cdot s^{(k-1)/k}\left(\int_0^s f'(\tau) f^{p-1}(\tau) d\tau\right)^{1/k} \times t^{(l-1)/l}\left(\int_0^t g'(\sigma) g^{q-1}(\sigma) d\sigma\right)^{1/l} \leq$$

$$\leq pq \frac{l \cdot s^{k-1}+h \cdot t^{l-1}}{hl}\left(\int_0^s f'(\tau) f^{p-1}(\tau) d\tau\right)^{1/k} \times \left(\int_0^t g'(\sigma) g^{q-1}(\sigma) d\sigma\right)^{1/l} \quad (20)$$

Dividing both sides of (20) by $\frac{l \cdot s^{k-1}+h \cdot t^{l-1}}{hl}$ and integrating over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using Holder integral inequality, we get that

$$\int_0^x \int_0^y \frac{hl\left(f^p(s)-f^p(0)\right)\left(g^q(t)-g^q(0)\right)}{l \cdot s^{k-1}+h \cdot t^{l-1}} dsdt \leq$$

$$\leq pq \int_0^x \left(\int_0^s \left(f'(\tau) f^{p-1}(\tau)\right)^k d\tau\right)^{1/k} ds \times \int_0^y \left(\int_0^t \left(g'(\sigma) g^{q-1}(\sigma)\right)^l d\sigma\right)^{1/l} dt \leq$$

$$\leq pqx^{\frac{k-1}{k}} \cdot y^{\frac{l-1}{l}} \left(\int_0^x \left(\int_0^s \left(f'(\tau) f^{p-1}(\tau) \right)^k d\tau \right) ds \right)^{\frac{1}{k}} \left(\int_0^y \left(\int_0^t \left(g'(\sigma) g^{q-1}(\sigma) \right)^l d\sigma \right) dt \right)^{\frac{1}{l}} =$$

$$= pq \cdot x^{\frac{k-1}{k}} \cdot y^{\frac{l-1}{l}} \left(\int_0^x (x-s) \left(f'(s) f^{p-1}(s) \right)^k ds \right)^{\frac{1}{k}} \left(\int_0^y (y-t) \left(g'(t) g^{q-1}(t) \right)^l dt \right)^{\frac{1}{l}}.$$

Remark 3. If we take $p = q = 1, f(0) = g(0) = 0$ in (17), then

$$\int_0^y \frac{f(s) g(t)}{l \cdot s^{k-1} + h \cdot t^{l-1}} dsdt \leq$$

$$\leq \frac{x^{(k-1)/k} \cdot y^{(l-1)/l}}{hl} \times \left(\int_0^x (x-s) \left(f'(s) \right)^k ds \right)^{1/k} \left(\int_0^y (y-t) \left(g'(t) \right)^l dt \right)^{1/l}. \quad (21)$$

This is just a new inequality similar to Theorem 2, which was given by B. G. Pachpatte in [8].

On the other hand, we apply the inequality (8) on the right-hand side of (17), we get that

$$\int_0^x \int_0^y \frac{hl \left(f^p(s) - f^p(0) \right) \left(g^q(t) - g^q(0) \right)}{l \cdot s^{k-1} + h \cdot t^{l-1}} dsdt \leq pqx^{(k-1)/k} \cdot y^{(l-1)/l} \times$$

$$\times \left(\frac{1}{h} \int_0^x (x-s) \left(f'(s) f^{p-1}(s) \right)^k ds + \frac{1}{l} \int_0^y (y-t) \left(g'(t) q^{q-1}(t) \right)^l dt \right). \quad (22)$$

Theorem 4. *Under the hypotheses of Theorem 3, if $f(s) \neq f(0)$, $g(t) \neq g(0)$ and let ϕ and ψ are two real-valued nonnegative, submultiplicative and monotone increasing convex functions defined on R_+, then*

$$\int_0^x \int_0^y \frac{hl \left(f(s) - f(0) \right) \left(g(t) - g(0) \right) \cdot \phi \left(\frac{f^p(s) - f^p(0)}{f(s) - (0)} \right) \psi \left(\frac{g^q(t) - g^q(0)}{g(t) - g(0)} \right)}{l \cdot s^{k-1} + h \cdot t^{l-1}} dsdt \leq \phi(p) \psi(q) \times$$

$$\times x^{(k-1)/k} \cdot y^{(l-1)/l} \left(\int_0^x (x-s) \left(f'(s) \phi \left(f^{p-1}(s) \right) \right)^k ds \right)^{1/k} \cdot$$

$$\cdot \left(\int_0^y (y-t) \left(g'(t) \left(q^{q-1}(t) \right) \right)^l dt \right)^{1/l} \quad (23)$$

Proof. From (18), we have

$$\frac{f^p(s)-f^p(0)}{f(s)-f(0)}=p\,\frac{\int_0^s f'(\tau)\,f^{p-1}(\tau)^k\,d\tau}{\int_0^s f'(\tau)}$$

From the hypotheses and using Jensen integral inequality and Holder integral inequality, we get that

$$\phi\left(\frac{f^p(s)-f^p(0)}{f(s)-f(0)}\right)=\phi\left(p\,\frac{\int_0^s f'(\tau)\,f^{p-1}(\tau)\,d\tau}{\int_0^s f'(\tau)}\right)\le\phi(p)\cdot\phi\left(\frac{\int_0^s f'(\tau)\,f^{p-1}(\tau)\,d\tau}{\int_0^s f'(\tau)}\right)\le$$

$$\le\frac{\phi(p)}{f(s)-f(0)}\int_0^s f'(\tau)\,\phi\left(f^{p-1}(\tau)\right)^k d\tau\le$$

$$\le\frac{\phi(p)}{f(s)-f(0)}s^{(k-1)/k}\left(\int_0^s\left(f'(\tau)\cdot\phi\left(f^{p-1}(\tau)\right)\right)^k d\tau\right)^{1/k}\qquad(24)$$

and similarly

$$\phi\left(\frac{g^q(t)-g^q(0)}{g(t)-g(0)}\right)\le\frac{\psi(q)}{g(t)-g(0)}s^{(l-1)/l}\left(\int_0^t\left(g'(\sigma)\cdot\psi\left(g^{q-1}(\sigma)\right)\right)^k d\sigma\right)^{1/l}\qquad(25)$$

Thus

$$\int_0^x\int_0^y\frac{hl\,(f(s)-f(0))\,(g(t)-g(0))\cdot\phi\left(\frac{f^p(s)-f^p(0)}{f(s)-(0)}\right)\psi\left(\frac{g^q(t)-g^q(0)}{g(t)-g(0)}\right)}{l\cdot s^{k-1}+h\cdot t^{l-1}}dsdt\le$$

$$\le\phi(p)\,\psi(q)\int_0^x\left(\int_0^s\left(f'(\tau)\cdot\phi\left(f^{p-1}(\tau)\right)\right)^k d\tau\right)^{1/k}ds\times$$

$$\times\int_0^y\left(\int_0^t\left(g'(\sigma)\cdot\psi\left(g^{q-1}(\sigma)\right)\right)^k d\sigma\right)^{1/l}dt\le$$

$$\le\phi(p)\,\psi(q)\,x^{(k-1)/k}\cdot y^{(l-1)/l}\left(\int_0^x\left(\int_0^s\left(f'(\tau)\cdot\phi\left(f^{p-1}(\tau)\right)\right)^k d\tau\right)ds\right)^{1/k}\times$$

$$\times\left(\int_0^y\left(\int_0^t\left(g'(\sigma)\cdot\psi\left(g^{q-1}(\sigma)\right)\right)^k d\sigma\right)dt\right)^{1/l}=$$

$$= \phi(p)\psi(q)x^{(k-1)/k}\cdot y^{(l-1)/l}\left(\int_0^x (x-s)\left(f'(s)\phi\left(f^{p-1}(s)\right)\right)^k ds\right)^{1/k}\times$$

$$\times\left(\int_0^y (y-t)\left(g'(t)\psi\left(q^{q-1}(t)\right)\right)^l dt\right)^{1/l}.$$

It is obvious that the inequality (23) can also reduces to the inequality (21).

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