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The Natural Proof of the Inequalities of Wallis Type

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1 Introduction

The formula of *Wallis*

$$\lim_{n\to\infty} W_n = \frac{\pi}{2}, \tag{1}$$

where $W_n = \dfrac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \ldots \cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot \ldots \cdot (2n-1)\cdot (2n+1)}$, is well-known and its proof is frequently given in the books of integral calculus.

At the same time, in the famous monograph of *D. S. Mitrinović* and *P. M. Vasič* "Analytic Inequalities" ([12], pag. 192), the authors present as an *inequality of Wallis* the following:

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n} < \frac{1}{\sqrt{\pi n}}. \tag{2}$$

In 1956, *D. K. Kazarinoff* has established in [9] a first refinement of the precedent inequality:

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n} < \frac{1}{\sqrt{\pi(n+1/4)}} \tag{3}$$

(this inequality also being mentioned in [10].)

Later, in 1985, *L.Panaitopol* has established in [13] the inequality:

$$\frac{4^n}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} < \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi\left(n+\frac{1}{4}\right)}},$$

which is equivalent with:

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} < \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}, \tag{4}$$

(because $\binom{2n}{n} = 4^n \cdot \dfrac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n}$) and refines (3).

In the mentioned paper [9] of *D. K. Kazarinoff*, the author uses for the proof of the inequality (3) some advanced analytic properties of the theory of the special functions, for instance the logarithmic derivative of the function ϕ, defined by the equality:

$$\Phi(\alpha)=\int_0^\pi \sin^\alpha x\,dx=\frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)},\ \ -1<\alpha<\infty,$$

a formula of *Legendre*

$$\frac{\Gamma'(\beta)}{\Gamma(\beta)}=-C+\int_0^1\frac{t^{\beta-1}-1}{t-1}\,dt,\ \ 0<\beta<\infty,$$

where C is the constant of *Euler*, namely $C=\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right)$ and other facts of same kind.

But it exists a possibility to give a certain simpler proof for the inequalities (2), (3) and (4) using a method closely related to the one introduced by *L. Panaitopol* in [13]. A difference between [13] and our paper (and also an improvement) is that in [13] the limits of type (9) are obtained using *Stirling*'s formula, while we obtain them using *Wallis*' formula, which is simpler as *Stirling*'s formula in all the theoretical treatises. Our aim in this paper is to present these proofs.

2 The simpler proofs of (2), (3) and (4)

Wanting a more convenient writing, we shall use the following notation:

$$\Omega_n=\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}\tag{5}$$

introduced by us in [16], which gives to (2), (3) and (4) a simpler form (also see in chronological order [17], [18], [19]).

From the obvious equality

$$W_n=\frac{1}{\Omega_n^2}\cdot\frac{1}{2n+1}\tag{6}$$

it follows

$$\Omega_n=\frac{1}{\sqrt{W_n}}\frac{1}{\sqrt{2n+1}},\tag{7}$$

and so

$$\Omega_n\sqrt{n}=\frac{1}{\sqrt{W_n}}\sqrt{\frac{n}{2n+1}}.\tag{8}$$

According to the formula (1) of *Wallis*, we obtain that the sequence of the right part of (8) is convergent, therefore the sequence of the left part also is convergent and we have

$$\lim_{n\to\infty} \Omega_n \sqrt{n} = \frac{1}{\sqrt{\pi}}. \tag{9}$$

Now, we can quickly prove the inequality (2). The left part is equivalent with the inequality $\frac{1}{\sqrt{\pi}} < \Omega_n \sqrt{n+\frac{1}{2}}$. Note $a_n = \Omega_n \sqrt{n+\frac{1}{2}}$. Because (9), we have $\lim_{n\to\infty} a_n = \frac{1}{\sqrt{\pi}}$. The sequence $(a_n)_n$ is strictly decreasing, because:

$$\frac{a_{n+1}}{a_n} = \frac{\Omega_{n+1}}{\Omega_n}\frac{\sqrt{n+3/2}}{\sqrt{n+1/2}} = \frac{2n+1}{2n+2}\frac{\sqrt{2n+3}}{\sqrt{2n+1}} = \frac{\sqrt{(2n+1)(2n+3)}}{2n+2} < 1.$$

Thus $\frac{1}{\sqrt{\pi}} < a_n$ and we have obtained the left part of (2).

The right part of (2) is equivalent with the inequality $\Omega_n < \frac{1}{\sqrt{\pi}}$. Note $b_n = \Omega_n \sqrt{n}$. We also have $\lim_{n\to\infty} b_n = \frac{1}{\sqrt{\pi}}$ (because (9)). The sequence $(b_n)_n$ is strictly increasing, because

$$\frac{b_{n+1}}{b_n} = \frac{\Omega_{n+1}}{\Omega_n}\frac{\sqrt{n+1}}{\sqrt{n}} = \frac{2n+1}{2n+2}\frac{\sqrt{n+1}}{\sqrt{n}} = \frac{2n+1}{\sqrt{2n(n+1)}} > 1.$$

Thus $b_n < \frac{1}{\sqrt{\pi}}$ and we have obtained the right part of (2).

To prove the right part of (3) and the left part of (4), we note $c_n = \Omega_n \sqrt{n+\frac{1}{4}}$ and $d_n = \Omega_n \sqrt{n+\frac{1}{4}+\frac{1}{32n}}$ and we work analogously: we observe that $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = \frac{1}{\sqrt{\pi}}$ and, using the ratio of two consecutive terms, we prove that the sequence $(c_n)_n$ is strictly increasing, respectively the sequence $(d_n)_n$ is strictly decreasing.

3 An explanation of the inequality (2) based on the Γ function

The limit of (9) shows us that the behavior of Ω_n is given in a first time the "approximative" equality $\Omega_n \simeq \frac{1}{\sqrt{\pi n}}$, or more exactly, considering (21),

$$\Omega_n = \frac{1}{\sqrt{\pi n}}(1+o(1)), \tag{10}$$

where $o(1)$ denote, as usually, an expression depending of the natural variable (a sequence) which tends to 0 for $n \to \infty$.

But this behavior also can be explained by some properties of the Γ function. Indeed, from the known formulae $\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi}\,(1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1))\cdot 2^{-n}$ and $\Gamma(n+1) = n!$, we obtain

$$\Omega_n = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}. \tag{11}$$

On the other hand there exists the following interesting inequality, proved by *I. Lazarević* and *A. Lupaş* ([10]) for a function $f : (0, \infty) \to (0, \infty)$ such that $\log f$ is a strictly convex function:

$$[A(x)]^{1-\theta} < \frac{f(x+1)}{f(x)} < [A(x+\theta)]^{1-\theta} \quad (0 < \theta < 1). \tag{12}$$

In particular, $\ln \Gamma$ is a strictly convex function1) and so, putting (as in [10]) $f = \Gamma$, then $A(x) = x$ and (12) can be written in the form

$$x^{1-\theta} < \frac{\Gamma(x+1)}{\Gamma(x+\theta)} < (x+\theta)^{1-\theta} \quad (x > 0,\ 0 < \theta < 1) \tag{13}$$

(when $x = n \in \mathbb{N}^*$ the inequality being discovered at first by *W. Gautschi* [7]; see also [5], [21]). For $x = n$ and $\theta = \dfrac{1}{2}$ we obtain:

$$\frac{1}{\sqrt{n+\frac{1}{2}}} < \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} < \frac{1}{\sqrt{n}}$$

and, according to (11), we find once again the inequality (2) of Wallis.

4 Addenda

a) Also there exists other refinements of the inequality (2), namely the following:

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{4(4n+3)}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}} \quad \text{(J.Gurland, 1956, [8])} \tag{14}$$

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{46n}\right)}} \quad \text{(L. Tóth, 1993, [15])} \tag{15}$$

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32(n+0,25)}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32(n+\theta)}\right)}};\ \theta > \frac{1}{4} \quad \text{([18])} \tag{16}$$

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[1] More, according to a well-known theorem of *H.Bohr* and *I. Mollerup* ([2],[1]), $\Gamma/(0,\infty)$ is the unique logarithmic-convex function, so that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$.

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}-\frac{1}{128n^2}\right)}}; \quad ([18])^{2)} \tag{17}$$

We also note an inequality of another kind:

$$\frac{1}{\sqrt{\pi}\sqrt[4]{n^2+\frac{n}{2}+\frac{1}{8}}} < \Omega_n < \frac{1}{\sqrt{\pi}\sqrt[4]{n^2+\frac{n}{2}+\frac{3}{32}}}; \tag{18}$$

R.E.Shafer, (1975) ([14]),
J.Grimland jr.(1977),
S.Glidewell (1977).

Its also can be refined as follows

$$\frac{1}{\sqrt{\pi}\sqrt[4]{n^2+\frac{n}{8}+0,125}} < \Omega_n < \frac{1}{\sqrt{\pi}\sqrt{n^2+\frac{n}{2}+0,124}}; \quad ([18]) \tag{19}$$

$$\frac{1}{\sqrt{\pi}\sqrt[4]{n^2+\frac{1}{8}}} < \Omega_n < \frac{1}{\sqrt{\pi}\sqrt[4]{n^2+\frac{n}{2}+\frac{1}{8}-\frac{1}{32n}}}; \quad ([18]) \tag{20}$$

b) An interesting explanation of the right part of the last three inequalities, based on the convexity, was given me by Professor *Alexandru Lupaş*, Ph. D. ([11]) and it is the following: According that $\log\Gamma$ is a strictly convex function, we have, for all strictly positive nodes x_i, $0 < x_1 < x_2 < \ldots < x_n$, $[x_1, x_2, x_3; \log\Gamma]$ [3] > 0, $[x_1, x_2, x_3, x_4; \log\Gamma] < 0$ etc. The last inequality gives successively:

$$-\frac{\log\Gamma(x_1)}{(x_2-x_1)(x_3-x_1)(x_4-x_1)} + \frac{\log\Gamma(x_2)}{(x_2-x_1)(x_3-x_2)(x_4-x_2)} -$$

$$-\frac{\log\Gamma(x_3)}{(x_3-x_1)(x_3-x_2)(x_4-x_3)} + \frac{\log\Gamma(x_4)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} < 0,$$

$$(\Gamma(x_2))^{\frac{1}{(x_2-x_1)(x_3-x_2)(x_4-x_2)}} \cdot (\Gamma(x_4))^{\frac{1}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}} <$$

$$< (\Gamma(x_1))^{\frac{1}{(x_2-x_1)(x_3-x_1)(x_4-x_1)}} \cdot (\Gamma(x_3))^{\frac{1}{(x_3-x_1)(x_3-x_2)(x_4-x_3)}},$$

$$(\Gamma(x_2))^{(x_4-x_1)(x_4-x_3)(x_3-x_1)} \cdot (\Gamma(x_4))^{(x_2-x_1)(x_3-x_1)(x_3-x_2)} <$$

$$< (\Gamma(x_1))^{(x_3-x_2)(x_4-x_3)(x_4-x_2)} \cdot (\Gamma(x_3))^{(x_2-x_1)(x_4-x_2)(x_4-x_1)}.$$

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[2] I have exposed these results (and also (23),(24)) at the first "Annual Meeting of the Romanian Society of Mathematical Sciences", Bucharest 1997, may 29-june 1, and I remember that I had the honor and the pleasure that my communication was seen by (in alphabetical order) Professor Cabiria Andreian Cazacu, Ph.D, Professor Petru Caraman, Ph.D. and Professor Constantin Corduneanu, Ph.D. (I think them for valuable discussions and suggestions given at that time). The next conferences of the Society was regularly helded so: 1998 in Cluj-Napoca, 1999 in Craiova, 2000 in Constanţa, 2001 in Braşov, 2002 in Sibiu. The next (2003) shall be held in Bistriţa, May 22-25.

[3] We denote so the divided differences.

Putting here $x_1 = n$, $x_2 = n + \dfrac{1}{2}$, $x_3 = n + 1$, $x_4 = n + \dfrac{3}{2}$, we obtain:

$$\left(\Gamma\left(n+\frac{1}{2}\right)\right)^{\frac{3}{2}\cdot\frac{1}{2}} \cdot \left[\Gamma\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right]^{\frac{1}{2}\cdot\frac{1}{2}} < \left(\frac{\Gamma(n+1)}{n}\right)^{\frac{1}{2}\cdot\frac{1}{2}} \cdot (\Gamma(n+1))^{\frac{1}{2}\cdot\frac{3}{2}},$$

i.e. $\dfrac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} < \dfrac{1}{\sqrt[4]{n\left(n+\frac{1}{2}\right)}}$. So $\Omega_n < \dfrac{1}{\sqrt{\pi}\sqrt[4]{n^2+\frac{n}{2}}}$.

c) All the successive refinements (14) - (17) of the evaluations of Ω_n has conducted us to the idea to give an asymptotic expansion of Ω_n, by a formula containing a square root. So, starting from a result of our paper [16]

$$\Omega_n = \frac{1}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \frac{21}{32768n^4} + \ldots\right) \quad {}^{4)} \tag{21}$$

(deducted in [16] from

$$W_n = \frac{\pi}{2}\left(1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2048n^4} - \frac{143}{8192n^5} + \ldots\right)\Bigg), \tag{22}$$

we have obtained in [18] the following asymptotic expansion:

$$\Omega_n = \frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32n} - \frac{1}{128n^2} - \frac{5}{2048n^3} + \frac{23}{4096n^4} + \ldots\right)}}. \tag{23}$$

We also have obtained (in relation with (18), (19), (20)) the asymptotic expansion:

$$\Omega_n = \frac{1}{\sqrt{\pi}\sqrt[4]{n^2 + \frac{n}{4} + \frac{1}{8} - \frac{1}{32n} - \frac{1}{256n^2} + \ldots}}. \tag{24}$$

d) Finally, another last remark: concerning the formula of Wallis (usually proved by Integral Calculus), we note that also there exists an interesting proof containing no integrals ([22]). But this proof is too much long and gives the impression that it was elaborated only as an answer to a pure scientific curiosity, *"pour l'honneur de l'ésprit humain"* using an expression of the well known french mathematician Jean Dieudonné. (This is a title of one of his books [4]).

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[4] Much later, in 1999, I have observed that an asymptotic expansion for $\binom{2n}{n}$ was already published in AMM, august-september, 1990, p. 626-630.

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