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Sufficient Conditions of Univalence for an Integral Operator

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Abstract. In this work we study an integral operator and determine conditions for the univalence of this integral operator.

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1 Introduction

Let A be the class of the functions f which are analytic in the unit disc $U=\{z\in C : |z|<1\}$ and $f(0)=f'(0)-1=0$.

We denote by S the class of the functions $f\in A$ which are univalent in U.

Petru T. Mocanu [1] defines the class of α convex functions M_α. For this class of functions we have the integral operator: $\mathcal{M}_\alpha : S^* \to M_\alpha$, $f=\mathcal{M}_\alpha(g)$, where S^* is the class of starlike functions and

$$f(z)=\left[\frac{1}{\alpha}\int_0^z g^{\frac{1}{\alpha}}(u)u^{-1}du\right]^\alpha,\ z\in U,\ \alpha>0. \tag{1}$$

If $g\in S^*$, then the function f is univalent in U.

2 Preliminary results

We need the following theorems.

Theorem 2.1. [3] *Let α be a complex number, $\operatorname{Re}\alpha>0$ and $f\in A$. If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\le 1 \tag{2}$$

for all $z\in U$, then the function

$$F_\alpha(z)=\left[\alpha\int_o^z u^{\alpha-1}f'(u)du\right]^{1/\alpha} \tag{3}$$

is in the class S.

Theorem 2.2. [2] *If the function g is regular in U, then for all $\xi\in U$ and $z\in U$ the following inequalities hold:*

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi-z}{1-\overline{z}\xi}\right|, \tag{4}$$

and

$$|g'(z)| \leq \frac{1-|g(z)|^2}{1-|z|^2}, \tag{5}$$

the equalities hold only in the case $g(z)=\epsilon\frac{z+u}{1+\overline{u}z}$ *where* $|\epsilon|=1$ *and* $|u|<1$.

Remark 2.3. [2] For $z=0$, from inequality (4)

$$\left|\frac{g(\xi)-g(0)}{1-\overline{g(0)}g(\xi)}\right| \leq |\xi| \tag{6}$$

and, hence

$$|g(\xi)| \leq \frac{|\xi|+|g(0)|}{1+|g(0)||\xi|}. \tag{7}$$

Considering $g(0)=a$ and $\xi=z$,

$$|g(z)| \leq \frac{|z|+|a|}{1+|a||z|}. \tag{8}$$

for all $z\in U$.

SCHWARZ Lemma. [2] *If the function* g *is regular in* U, $g(0)=0$ *and* $|g(z)|\leq 1$ *for all* $z\in U$, *then the following inequalities hold*

$$|g(z)| \leq |z|, \tag{9}$$

for all $z\in U$, *and* $|g'(0)|\leq 1$, *the equalities (in inequality (9)) for* $z\neq 0$ *hold only in the case* $g(z)=\epsilon z$, *where* $|\epsilon|=1$.

3 Main results

Theorem 3.1. *Let* α *be a complex number with* $\operatorname{Re}\alpha=1$ *and* $g\in S$, $g(z)=z+a_2z^2+\cdots$. *If*

$$\left|\frac{zg'(z)}{g(z)}-1\right| \leq 1 \tag{10}$$

for all $z\in U$, *then the function*

$$T_\alpha(z)=\left[\frac{1}{\alpha}\int_0^z g^{\frac{1}{\alpha}}(u)u^{-1}du\right]^\alpha \tag{11}$$

is in the class S.

Proof. Let us note $\frac{1}{\alpha}=\beta$. We observe that

$$T_{\frac{1}{\beta}}(z)=\left[\beta\int_0^z u^{\beta-1}\left(\frac{g(u)}{u}\right)^\beta du\right]^{1/\beta}. \tag{12}$$

Let us consider the function

$$f(z)=\int_0^z\left(\frac{g(u)}{u}\right)^\beta du. \tag{13}$$

The function

$$p(z)=\frac{1}{|\beta|}\frac{zf''(z)}{f'(z)}, \tag{14}$$

is regular in U and because $\operatorname{Re}\alpha=1$ it results that

$$|\beta| \le 1. \tag{15}$$

From (13) and (14) it follows that

$$p(z) = \frac{\beta}{|\beta|}\left[\frac{zg'(z)}{g(z)} - 1\right]. \tag{16}$$

Using (16) and (10) we have

$$|p(z)| \le 1 \tag{17}$$

for all $z \in U$. For (16) we obtain $p(0) = 0$ and applying Schwarz-Lemma we have

$$\frac{1}{|\beta|}\left|\frac{zf''(z)}{f'(z)}\right| \le |z| \tag{18}$$

for all $z \in U$, and hence, we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le |\beta|\left(1-|z|^2\right)|z|. \tag{19}$$

Because $\max_{|z|\le 1}\left(1-|z|^2\right)|z| = \frac{2}{3\sqrt{3}}$, from (19) and (15) we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{20}$$

for all $z \in U$. From (20), (13), (12) and Theorem 2.1 for $\operatorname{Re}\alpha = 1$ it follows that T_α is in the class S.

Theorem 3.2. *Let α be a complex number with $\operatorname{Re}\alpha > 0$ and the function $g \in S, g(z) = z + a_2z^2 + \cdots$. If*

$$\left|\frac{zg'(z) - g(z)}{zg(z)}\right| \le 1 \tag{21}$$

for all $z \in U$ and

$$|\alpha| \ge \max_{|z|\le 1}\left[\frac{1-|z|^{2\operatorname{Re}\frac{1}{\alpha}}}{\operatorname{Re}\frac{1}{\alpha}}|z|\frac{|z|+|a_2|}{1+|a_2||z|}\right] \tag{22}$$

then the function $T_\alpha(z)$ defines by (11) is in the class S.

Proof. Let us note $\frac{1}{\alpha} = \beta$. We have

$$T_{\frac{1}{\beta}}(z) = \left[\beta\int_o^z u^{\beta-1}\left(\frac{g(u)}{u}\right)^\beta du\right]^{1/\beta}. \tag{23}$$

Let's consider the function

$$f(z) = \int_o^z \left(\frac{g(u)}{u}\right)^\beta du. \tag{24}$$

The function f is regular in U. Let the function

$$h(z) = \frac{1}{|\beta|}\frac{f''(z)}{f'(z)}, \tag{25}$$

where the constant $|\beta|$ satisfies the inequality

$$|\beta| \le \frac{1}{\max_{|z|\le 1}\left[\frac{1-|z|^{2\operatorname{Re}\beta}}{\operatorname{Re}\beta}|z|\frac{|z|+|a_2|}{1+|a_2||z|}\right]} \tag{26}$$

The function h is regular in U. From (25) and (24) we obtain

$$h(z) = \frac{\beta}{|\beta|}\left[\frac{zg'(z) - g(z)}{zg(z)}\right]. \tag{27}$$

and using the inequality (21) we have

$$|h(z)| < 1, \tag{28}$$

for all $z \in U$ and $|h(0)| = |a_2|$. Applying Remark 2.3 we have

$$\left|\frac{1}{\beta}\frac{f''(z)}{f'(z)}\right| \le \frac{|z| + |a_2|}{1 + |a_2||z|} \tag{29}$$

for all $z \in U$. From (29) it results that

$$\left(\frac{1 - |z|^{2\operatorname{Re}\beta}}{\operatorname{Re}\beta}\right)\left|\frac{zf''(z)}{f'(z)}\right| \le |\beta|\frac{1 - |z|^{2\operatorname{Re}\beta}}{\operatorname{Re}\beta}|z|\frac{|z| + |a_2|}{1 + |a_2||z|}. \tag{30}$$

for all $z \in U$.

Let us consider the function $Q : [0, 1] \to \Re$, $Q(x) = \frac{(1-x^{2\operatorname{Re}\beta})}{\operatorname{Re}\beta}x\frac{x+|a_2|}{1+|a_2|x}$, $x = |z|$. Because $Q\left(\frac{1}{2}\right) > 0$ it results that

$$\max_{x\in[0,1]} Q(x) > 0. \tag{31}$$

Using this result, from (30) we conclude

$$\frac{1 - |z|^{2\operatorname{Re}\beta}}{\operatorname{Re}\beta}\left|\frac{zf''(z)}{f'(z)}\right| \le |\beta|\max_{|z|\le 1}\left[\frac{1 - |z|^{2\operatorname{Re}\beta}}{\operatorname{Re}\beta}|z|\frac{|z| + |a_2|}{1 + |a_2||z|}\right]. \tag{32}$$

From (32) and (26) we obtain

$$\frac{1 - |z|^{2\operatorname{Re}\beta}}{\operatorname{Re}\beta}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{33}$$

for all $z \in U$.

From (33) and Theorem 2.1, we obtain that the function

$$G_{\frac{1}{\beta}}(z) = \left[\beta\int_0^z u^{\beta-1}f'(u)du\right]^{1/\beta} \tag{34}$$

belongs to the class S.

From (34) and (24) we have $T_{\frac{1}{\beta}}$ is in the class S, and hence, we conclude that the function T_α belongs to the class S.

Observation. In the Theorem 3.2 we obtain the conditions of univalence which use the coeficient a_2 too.

References

[1] Mocanu, P. T., *Une propriete de convexite generalise dans la theorie de la representation conforme,* Mathematica (Cluj), 11 (34), 1(1969), 127-133.

[2] Nehari, Z., *Conformal mapping,* Mc Graw-Hill Book Comp., New York, Toronto, London, 1952.

[3] Pascu, N. N., *On a univalence criterion II,* Inter. Seminar on Functional Equations Approximation and Convexity, Cluj-Napoca, Preprint No. 6, 1985, 153-154.

[4] Pescar, V., *New univalence criteria*, "Transilvania" University of Braşov, 2002.

[5] Pommerenke, C., *Univalent functions*, Vandenhoeck Ruprecht in Gottingen, 1975.