LIBERTAS MATHEMATICA, vol XXIV (2004)

# Capacities and Markov Processes

**Manuela L. BUJORIANU**

**Abstract.** We present the capacity associated to a product of Markov processes. We, also, give an estimation, based on Martin capacity, of the probability of the reach sets for a Markov process with some "nice" properties.

## 1 Introduction

The concept of capacity was initially introduced to respond to a practical necessity and it proved to be a far more useful tool for certain non-linear phenomena than measures were. However, its implications are more far-reaching than it would first appear and it is the purpose of this paper to explore some of these aspects.

The first part of this analysis considers the capacities associated to the product of two standard Markov processes with some nice properties. The working definition for capacity associated to a standard Markov process is that provided by [19]. It has already been established the correlation between a standard Markov process and a (non)symmetric quasi-regular Dirichlet form (see [24]) so our contribution is to construct in fact a capacity associated to the product of two Dirichlet forms. This approach is somewhat different than that of the initial mainstream (see [2]) as it expresses the capacities of product of two Borel sets as a formula of the factor capacities.

Furthermore, a different type of capacity associated to a Markov process, namely the Martin capacity, is met in a particular situation entailed by the stochastic hybrid systems (see [8]) and this makes the subject of the second part of this paper. For certain stochastic hybrid systems we can develop some models of different types of Markov processes. In a probabilistic framework, the reachability problem consists of determining the probability that the system trajectories enter some prespecified set starting from a certain set of initial conditions with a given probability distribution. The methodology for the reachability analysis of stochastic hybrid systems is in itself a difficult problem involving a theoretical aspect of the measurability of the reachability sets (see [8]) as well as a computational issue regarding the estimation of the reach set probability. It is this second aspect that is further explained in this paper while the basis for the estimation of the reach set probability is the Martin capacity.

The study of capacities inserted in the framework of Markov processes have proved to be very useful for their practical implications. For instance, the results we reached in the first part of this work form the initial steps for future works regarding the product of two phenomena modelled by Markov processes.

## 2 Capacity of a Transient Markov Process

Throughout the paper $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P_x)$ and $\widehat{X} = (\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{F}}_t, \widehat{\theta}_t, \widehat{X}_t, \widehat{P}^x)$ will be standard Markov processes in weak duality with respect to a $\sigma$-finite measure $m$ on a locally compact second countable state space $(\mathbb{S}, \mathcal{B})$. This means that

- $X$ and $\widehat{X}$ are strong Markov processes with path that are right-continuous with left limits up to their respective lifetimes $\zeta$ and $\widehat{\zeta}$,
- $X$ and $\widehat{X}$ are quasi-left-continuous on $[0, \zeta)$ and $[0, \widehat{\zeta})$ respectively,
- the transition operator semigroups $(P_t)_{t \geq 0}$ and $(\widehat{P}_t)_{t \geq 0}$ of $X$ and $\widehat{X}$ map $b\mathcal{B}$ into itself;
- for all positive $\mathcal{B}$-measurable functions $f$ and $g$

$$\int f P_t g \, dm = \int \widehat{P}_t f g \, dm, \quad t > 0. \tag{1}$$

It follows that the resolvent operators $U^\alpha := \int_0^\infty e^{-\alpha t} P_t \, dt$, $\widehat{U}^\alpha := \int_0^\infty e^{-\alpha t} \widehat{P}_t dt$, $\alpha \geq 0$, preserve Borel measurability and are dual in a sense analogous to (1). We write $U$ for $U^0$ and $\widehat{U}$ for $\widehat{U}^0$. Basic references for standard processes in duality are Blumenthal and Getoor [5].

The measure $P_x$ (resp. $\widehat{P}_x$) is the law of $(X_t)_{t \geq 0}$ (resp. $(\widehat{X}_t)_{t \geq 0}$) under the initial condition $X_0 = x$ (resp. $\widehat{X}_0 = x$). A cemetery state $\Delta$ is adjointed to $\mathbb{S}$, as an isolated point if $\mathbb{S}$ is compact and as the point at infinity otherwise. The lifetime $\zeta := \inf\{t > 0 : X_t = \Delta\}$ accounts the possibility $P_t 1(x) < 1$, in that $P_x(t < \zeta) = 1 - P_t 1(x)$. We adhere to the convention that a function (resp. measure) defined on $\mathbb{S}$ (resp. $\mathcal{B}^*$) is extended to $\mathbb{S}_\Delta := \mathbb{S} \cup \{\Delta\}$ (resp. $\mathcal{B}^* \vee \{\Delta\}$) by declaring its value at $\Delta$ (resp. $\{\Delta\}$) to be 0. We will use the prefix "co-" to signal objects or notions defined with respect to $\widehat{X}$ (e.g. a co-excessive function is an excessive function of $\widehat{X}$).

We assume that $X$ and $\widehat{X}$ are transient. This means that there are strictly Borel functions $q$ and $\widehat{q}$ such that $Uq$ and $\widehat{U}\widehat{q}$ are bounded functions (i.e. the potential kernels are assumed to be proper kernels).

Before introducing our problem, we need to recall the definition of the capacity $Cap^X$. An $\mathcal{B}^*$−function $f : \mathbb{S} \to [0, \infty]$ is excessive provided $t \mapsto P_t f(x)$ is decreasing and right-continuous on $[0, \infty)$ for each $x \in \mathbb{S}$. Let denote by $\mathcal{E}$ the cone of excessive functions.

For a Borel set $B \subset \mathbb{S}$ let $T_B := \inf\{t > 0 | X_t \in B\}$ denote the hitting time of $B$ by $X$, and let $P_B$ denote the associated hitting operator, i.e. $P_B := P_x(f(X_t) : T_B < \infty)$, $f \in \mathcal{B}$, $f \geq 0$. Recall that $T_B$ is an optimal (stopping) time and $P_B$ maps $\mathcal{E}$ into $\mathcal{E}$ (see [5]). In particular

$$\varphi_B := P_B 1(x) = P_x(T_B < \infty) = P_x(T_B < \zeta) \tag{2}$$

is an excessive function for any $B \in \mathcal{B}$.

Let $Exc$ and $\widehat{Exc}$ denote the cones of excessive and coexcessive measures. Recall that $\xi \in Exc$, for example, means that $\xi$ is a $\sigma$-finite measure on $\mathbb{S}$ with $\xi P_t \leq \xi$ for all $t > 0$. An excessive measure of the form $\nu U$ ($\nu$ a measure on $\mathbb{S}$) is called *potential*. To introduce the capacity one has to define as in [19] the energy functional $L$. For $h$ excessive and $\xi \in Exc$, one defines

$$L(\xi, h) := \sup\{\nu(h) : \nu U \in Exc, \ \nu U \leq \xi\}.$$

We record from [19] the following properties of $L$:

(F1) $L(\nu U, h) = \nu(h)$, $L(\xi, Uf) = \xi(f)$;

(F2) $h_n \uparrow h \Rightarrow L(\xi, h_n) \uparrow L(\xi, h)$;

(F3) $\xi_n \uparrow h \Rightarrow L(\xi_n, h) \uparrow L(\xi, h)$.

Since $X$ is transient, Hunt's balayage operation on $Exc$ can be defined as follows. Given $\xi \in Exc$ there exists a sequence $(\mu_n)$ of finite measures on $\mathbb{S}$ such that $\mu_n U \uparrow \xi$. The limit

$$R_B\xi = \uparrow \mu_n P_B U$$

is an excessive measure dominated by $\xi$ and is independent of the particular approximating sequence $(\mu_n U)$ (see [15]). From [19] we know that

$$L(R_B\xi, h) = L(\xi, P_B h).$$

The capacity $Cap^X$ is now defined by

$$Cap^X(X) = L(m, P_B 1) = L(R_B m, 1).$$

It is proved in [19] that $Cap^X$ is increasing, strongly subadditive, and ascending. In particular, $Cap^X$ is countably subadditive. Reversing the roles of $X$ and $\widehat{X}$ one obtains a dual capacity $Cap^{\widehat{X}}$, but by Hunt's switching identity and standardness imply that it coincides with $Cap^X$ (see [16]). Furthermore, in the symmetric case $Cap^X$ coincides with the (fine) 0-order capacity defined in terms of the associated Dirichlet forms (see [15]).

Using the properties (F1)-(F3) of the energy functional $L$ one can prove that

$$Cap^X(B) = \uparrow \lim_n \mu_n \varphi_B, \quad B \in \mathcal{B}$$

where $(\mu_n U)$ is an increasing sequence of potentials with setwise limit $m$ (see [16]). Notice that for any excessive function $h$ the integral $\mu_n(h)$ increases with $n$ to a limit whose value is independent of the specific choice of $(\mu_n U)$ approximating $m$.

**Remark 1.** [15] If $B$ is open and $Cap^X(B) < \infty$ then $\varphi_B$ is exactly the 0-order equilibrium potential of $B$.

## 3 Capacity of Product Process

Let $X^{(1)} = (X_t^{(1)}, P_x)$ and $X^{(2)} = (X_t^{(2)}, P_x)$ be standard Markov processes with the state space $\mathbb{S}^{(1)}$, respectively $\mathbb{S}^{(2)}$. $\mathbb{S}^{(1)}$, $\mathbb{S}^{(2)}$ are supposed locally compact second countable state spaces. Let $m_1$, $m_2$ two $\sigma$-finite measures defined on the Borel $\sigma-$algebra of $\mathbb{S}^{(1)}$, respectively $\mathbb{S}^{(2)}$. Let $\widehat{X}^{(1)}$ (resp. $\widehat{X}^{(2)}$) the dual process of $X^{(1)}$ (resp. $X^{(2)}$) with respect $m_1$ (resp. $m_2$). We shall denote: $\mathbb{S} = \mathbb{S}^{(1)} \times \mathbb{S}^{(2)}$, $\mathcal{B} = \mathcal{B}^{(1)} \otimes \mathcal{B}^{(2)}$.

For any functions $f_1 : \mathbb{S}^{(1)} \to \mathbb{R}$, $f_2 : \mathbb{S}^{(2)} \to \mathbb{R}$, we denote by $f_1 \otimes f_2$ the function defined on $\mathbb{S}$ by: $(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1) f_2(x_2)$. This notation extends to functions with values in $[0, \infty]$ with convention: $0 \cdot \infty = 0$. If $\mu_1$ and $\mu_2$ are measures on $(\mathbb{S}^{(1)}, \mathcal{B}^{(1)})$, $(\mathbb{S}^{(2)}, \mathcal{B}^{(2)})$ then $\mu_1 \otimes \mu_2$ will denote the product measure on $(\mathbb{S}, \mathcal{B})$. Hence: $(\mu_1 \otimes \mu_2)(f_1 \otimes f_2) = \mu_1(f_1)\mu_2(f_2)$.

To each kernel $U_1$ on $(\mathbb{S}^{(1)}, \mathcal{B}^{(1)})$ we associate a kernel $\widetilde{U}_1$ on $(\mathbb{S}, \mathcal{B})$ defined by: $\widetilde{U}_1 f(x_1, x_2) = U_1 f(\cdot, x_2)(x_1)$. In a similar way, to each kernel $U_2$ on $(\mathbb{S}^{(2)}, \mathcal{B}^{(2)})$ we can associate a kernel $\widetilde{U}_2$ on $(\mathbb{S}, \mathcal{B})$ defined by: $\widetilde{U}_2 f(x_1, x_2) = U_2 f(x_1, \cdot)(x_2)$. One can prove that $\widetilde{U}_1 \circ \widetilde{U}_2 = \widetilde{U}_2 \circ \widetilde{U}_1$ (see [26]). The tensor product of kernels $U_1$ and $U_2$ is defined as follows:

$$U_1 \otimes U_2 = \widetilde{U}_1 \circ \widetilde{U}_2 = \widetilde{U}_2 \circ \widetilde{U}_1.$$

We shall use the following notation: $U_1 \oplus U_2 = \widetilde{U}_1 + \widetilde{U}_2$.

Let $(P_t^1)$, $(P_t^2)$ be the semigroups associated to $X^{(1)}$, $X^{(2)}$. The associated resolvent will be denoted by: $(U_1^\alpha)_{\alpha>0}$ and $(U_2^\alpha)_{\alpha>0}$. Let us define $\widetilde{P_t^1}$ and $\widetilde{P_t^2}$ by

$$\widetilde{P_t^1} f(x,y) = P_t^1(f(\cdot,y))(x); \quad \widetilde{P_t^2} f(x,y) = P_t^2(f(x,\cdot))(y)$$

where $f \in \mathcal{B}$.

**Proposition 1.** [26] *The semigroup associated to the product process* $X$ *of* $X^{(1)}$ *with* $X^{(2)}$ *is* $P_t = \widetilde{P_t^1}\ \widetilde{P_t^2} = \widetilde{P_t^2}\widetilde{P_t^1}$.

**Proposition 2.** [26] *The resolvent* $(U^\alpha)_{\alpha>0}$ *associated with the product semigroup is related to the factor resolvent by the following formula:*

$$(U_1^\alpha \oplus U_2^\beta) \circ U^{\alpha+\beta} = U^{\alpha+\beta} \circ (U_1^\alpha \oplus U_2^\beta) = U_1^\alpha \otimes U_2^\beta, \quad \forall \alpha > 0,\ \beta > 0. \tag{3}$$

*Moreover, if* $U_1^0$ *is proper (resp. bounded), then* $U^0$ *is proper (resp. bounded).*

**Remark 2.** A similar formula of (3) can be written for the dual process $\widehat{X}^{(1)}$, $\widehat{X}^{(2)}$, $\widehat{X}$.

**Corollary 3.** *Formula (3) gives the relation between the kernel operators* $U_1$, $U_2$, $U$ *of the processes considered:*

$$(U_1 \oplus U_2) \circ U = U \circ (U_1 \oplus U_2) = U_1 \otimes U_2. \tag{4}$$

*Moreover, if at least one process* $X^{(1)}$ *or* $X^{(2)}$ *is transient then the product process is transient.*

We suppose now that the two standard processes $X^{(1)}$ and $X^{(2)}$ are transient. Then there are two sequences $(\mu_n^{(i)})$, $i = 1,2$ of measures defined on $(\mathbb{S}^{(i)}, \mathcal{B}^{(i)})$, $i = 1,2$ such that $U_i\mu_n^{(i)}$, $i = 1,2$ is an increasing sequence of potentials with setwise limit $m_i$, $i = 1,2$. Therefore, formula (4) applied for $\mu_n^{(1)} \otimes \mu_n^{(2)}$ gives that $(m_1 \otimes \mu_n^{(2)} + \mu_n^{(1)} \otimes m_2)U$ is an increasing sequence of potentials with setwise limit $m_1 \otimes m_2 = m$.

Let $B \in \mathcal{B}$ and $T_B$ (resp. $P_B$) the hitting time (resp. the hitting operator) with respect to the product process $X$. Then the capacity $Cap^X$ of $B$ can be defined as

$$Cap^X(B) = \uparrow \lim_n (m_1 \otimes \mu_n^{(2)} + \mu_n^{(1)} \otimes m_2)(\varphi_B)$$

where $\varphi_B$ is given by (2).

Let us denote by $||\cdot||_i$, $i = 1,2$ the norm of $L^1(\mathbb{S}^{(i)}, m_i)$, $i = 1,2$.

**Proposition 4.** *Let* $B_1 \in \mathcal{B}^{(1)}$ *and* $B_2 \in \mathcal{B}^{(2)}$. *The capacity* $Cap^X(B_1 \times B_2)$ *with the product process* $X$ *is related to the factor capacities by the following formula:*

$$Cap^X(B_1 \times B_2) = Cap^{X_2}(B_2)||\varphi_{B_1}^{(1)}||_1 + Cap^{X_1}(B_1)||\varphi_{B_2}^{(2)}||_2.$$

**Proof.** Let denote $B_1 \times B_2$ by $B$. For $f_1 \in \mathcal{B}^{(1)}$, $f_2 \in \mathcal{B}^{(2)}$ two positive measurable function from the tensor product kernel definition we obtain

$$P_B(f_1 \otimes f_2)(x_1, x_2) = (P_{B_1} \otimes P_{B_2})(f_1 \otimes f_2)(x_1, x_2) = P_{B_1} f_1(x_1) P_{B_2} f_2(x_2).$$

Therefore $\varphi_B(x_1, x_2) = \varphi_B^{(1)}(x_1)\varphi_B^{(2)}(x_2)$. Then

$$\begin{aligned}
Cap^X(B) &= \uparrow \lim_n (m_1 \otimes \mu_n^{(2)} + \mu_n^{(1)} \otimes m_2)(\varphi_B) = \\
&= \uparrow \lim_n \left\{ \int \varphi_B(x_1, x_2) m_1 \otimes \mu_n^{(2)}(dx_1, dx_2) + \int \varphi_B(x_1, x_2) \mu_n^{(1)} \otimes m_2(dx_1, dx_2) \right\} = \\
&= \uparrow \lim_n \left\{ \int \varphi_B^{(1)}(x_1) m_1(dx1) \int \varphi_B^{(2)}(x_2) \mu_n^{(2)}(dx_2) + \right. \\
&\quad \left. + \int \varphi_B^{(1)}(x_1) \mu_n^{(1)}(dx_1) \int \varphi_B^{(2)}(x_2) m_2(dx_2) \right\} = \\
&= Cap^{X_2}(B_2) ||\varphi_{B_1}^{(1)}||_1 + Cap^{X_1}(B_1) ||\varphi_{B_2}^{(2)}||_2.
\end{aligned}$$

# 4 Computation of Reach Set Probability

Let $\mathbb{M}$ be a Markov process with CADLAG property. Let $x$ be a given state in $\mathbb{S}$ and $E$ a Borel set in $\mathbb{S}$. We are interested to estimate

$$P_x(T_E < T) \quad \text{and} \quad P_x(T_E < \infty) \tag{HP}$$

where $T > 0$ is fixed. One might observe that it is enough to treat only the estimation problem of $P_x(T_E < \infty)$, because the estimation problem of $P_x(T_E < T)$ can be reduced to the first problem if we consider the "killed process" after time $T$ (see e.g. [5]).

## 4.1 Some History

In the Markov process literature we can find very nice formulas for $P_x(T_E < \infty)$. For example, if the underlying process is the Brownian motion $\{X_t, t \geq 0\}$ in $\mathbb{R}^3$; $P_x(\cdot)$ denotes the probability (Wiener measure) when all paths issue from the point $x$; $E$ is a compact set (the conductor body); $T_E = T_E(\omega)$ is the hitting time of $E$ by the path $\omega$ then

$$P_x(T_E < \infty) = \int_{\partial E} g(x, y) \mu_E(dy) \tag{5}$$

where $\partial E$ is the boundary of $E$; $g$ $(x, y)$ is the associated *potential density*

$$g(x, y) = \frac{1}{2\pi |x - y|};$$

and $\mu_E$ is called *equilibrium measure.*

On the other hand, the probability that a Brownian motion will ever visit a given set $E$ which appears in the left hand side of (5) can be classically estimated using the capacity of $E$ with respect to the Green kernel $g(x, y)$. In [1] it is shown that replacing the Green kernel by the Martin kernel $g(x, y)/g(0, y)$ yields improved estimates, which are exact up a factor of 2.

**Proposition 5.** *Let $\{B_d(t)\}$ denote standard d-dimensional Brownian motion with $B_d(0) = 0$ and $d \geq 3$. Let $E$ be any closed set in $\mathbb{R}^d$. Then*

$$\frac{1}{2} Cap_K(E) \leq \mathbf{P}[\exists t > 0 : B_d(t) \in E] \leq Cap_K(E)$$

*where*

$$K(x,y) = \frac{||y||^{d-2}}{||x-y||^{d-2}}$$

*for* $x \neq y$ *in* $\mathbb{R}^d$ *and* $K(x,x) = \infty$. *Here* $||x-y||$ *is the Euclidean distance and*

$$Cap_K(E) = \left[ \inf_{\mu(E)=1} \int_E \int_E K(x,y) d\mu(x) d\mu(y) \right]^{-1}.$$

Kai Lai Chung in [9] extended the formula (5) for temporally homogeneous transient Markov processes $\{X_t, t \geq 0\}$, taking values in a topological space $\mathbb{S}$ which is locally compact and has a countable base with its Borel $\sigma$-algebra $\mathcal{B}(\mathbb{S})$. The processes have also the CADLAG property. It is natural to put the problem of generalization of above Brownian motion result to more general Markov processes, using the Kai Lai Chung's result.

## 4.2 Hypotheses

For an estimation of $(HP)$ for a general class of Markov processes, in an analogous way as in Prop. 5, we start with a generalization of (5) proved in [9]. We have to consider the hypotheses from [9]: the underlying process is a temporally homogeneous Markov process $\{X_t, t \geq 0\}$, taking values in a locally compact space $\mathbb{S}$ with a countable base endowed with its Borel $\sigma$-algebra $\mathcal{B}(\mathbb{S})$. For the aim of this paper, it is enough to suppose that $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ is a Borel space. The transition semigroup is assumed to be Borelian. To get (5) it is not necessary to suppose the process is a standard or Hunt process; it is sufficient to assume that all paths are right continuous and have left limits in the interval $[0, \infty)$, i.e. the process has the CADLAG property (see [9]).

Let $p(t,x,B)$, $t > 0, x \in \mathbb{S}, B \in \mathcal{B}(\mathbb{S})$ the transition function associated to the given Markov process.

**Assumption 1.** All the measures $p(t,x,\cdot)$ are *absolutely continuous* with respect to a $\sigma$-finite measure $m$ on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$.

We denote the Radon-Nycodim derivative of $p(t,x,\cdot)$ by $\rho_t(x,\cdot)$, i.e.,

$$\rho_t(x,y) = p(t,x,dy)/m(dy).$$

This can be chosen (see [11]) to be measurable in $x, y$ and to satisfy the relation.

$$\int_{\mathbb{S}} \rho_s(x,y) m(dy) \rho_t(y,\nu) = \rho_{t+s}(x,\nu).$$

A $\sigma$-finite measure $m$ on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ is called *reference measure* if $m(B) = 0$ if and only if $p(t,x,B) = 0$ for all $t$ and $x$. Throughout this paper we suppose that $m$, in the absolutely continuity assumption, is a reference measure.

**Remark 3.** [5] If all $p-$excessive functions $(p > 0)$ associated to the underlying semigroup are lower semicontinuous then there exists a reference measure

$$m = \sum_{n=1}^{\infty} \frac{1}{2^n} \varepsilon^{\{y_n\}}$$

where $\{y_n\}$ is a countable dense subset of $\mathbb{S}$ and $\varepsilon^{\{y_n\}}$ are Dirac measures.

We define the *Green measure* $g_x$, $x \in \mathbb{S}$ as $g_x(B) = \int_0^\infty p(t,x,B)dt$, which is also the expectation $E^x(\eta_B)$ of the *occupancy time of* $B \in \mathcal{B}(\mathbb{S})$, defined $\eta_B := \int_0^\infty 1_{\{X_t \in B\}} dt$. In the same manner we define the *Green kernel*

$$g(x,y) = \int_0^\infty \rho_t(x,y)dt.$$

It is clear that, if $(P_t)$ is the semigroup of operators associated to the given Markov process, then we get

$$P_t\varphi(x) = E^x\varphi(X_t) = \int_{\mathbb{S}} \rho_t(x,y)\varphi(y)m(dy)$$

and the *kernel operator*

$$G\varphi(x) = E^x\{\int_0^\infty \varphi(X_t)dt\} = \int_{\mathbb{S}} g(x,y)\varphi(y)m(dy)$$

where $\varphi$ is any positive Borel measurable function.

A Borel set $E$ of $\mathbb{S}$ is called *transient* iff for almost every path $\omega$, there is a finite time $t^*(\omega)$ such that $X_t(\omega) \notin E$ for $t > t^*(\omega)$. Define

$$Reach_\infty(E) = \{\omega \in \Omega | \exists t > 0 : X_t(\omega) \in E\}$$

$$\gamma_E(\omega) = \begin{cases} sup\{t > 0 | X_t(\omega) \in E\} & \text{if } \omega \in Reach_\infty(E) \\ 0 & \text{if } \omega \in \Omega \backslash Reach_\infty(E) \end{cases}$$

Then $E$ is transient if and only if $\gamma_E < \infty$ a.s. (almost surely). $\gamma_E$ is a random variable, called the *last exit time* from $E$. The last exit time does not belong to the standard equipment, because it is not a stopping time (see [10] for more comments). It then follows that

$$x_{\gamma_E -} \in \overline{E} \text{ a.s.}$$

The proof of analogous formula to (5) for Markov processes, in [9], is dealing with two important notions:

- the *distribution of the last exit position* $x_{\gamma_E -}$

$$L^E(x,A) = P_x(\gamma_E > 0; x_{\gamma_E -} \in A), x \in \mathbb{S}, A \in \mathcal{B}(\mathbb{S})$$

- the *kernel operator* $G$.

In [9] the following assumptions on the Green kernel $g$ are necessary to prove the desired formula:

**Assumption 2.** *i*) $y \to g(x,y)^{-1}$ is finite continuous, for $x \in \mathbb{S}$;

*ii*) $g(x,y) = +\infty$ if and only if $x = y$.

Under these assumptions, there exists an equilibrium measure $\mu_E$, which is $\sigma$–finite, concentrated in $\overline{E}$ such that

$$\forall x \in \mathbb{S} : \mu_E(dy) = L^E(x,dy)g(x,y)^{-1}.$$

For a transient set $E$ we have

$$\{0 \leq T_E < \infty\} = \{0 < \gamma_E < \infty\}.$$

The final result is, for each Borel set $A \subset \overline{E}$, and each $x \in \mathbb{S}$:

$$P_x(x_{\gamma_E -} \in A) = L^E(x,A) = \int_A g(x,y)\mu_E(dy)$$

in particular,

$$P_x(T_E < \infty) = L^E(x, \overline{E}) = \int_{\overline{E}} g(x,y)\mu_E(dy). \tag{6}$$

## 4.3 Main Results

Let $\Lambda$ be a set and $\mathcal{B}$ a $\sigma$-algebra of subsets of $\Lambda$. Given a measurable function $F : \Lambda \times \Lambda \to [0, \infty]$ and a finite measure $\mu$ on $(\Lambda, \mathcal{B})$, the *F-energy* of $\mu$ is

$$F(\mu) = F(\mu, \mu) = \int_\Lambda \int_\Lambda F(x,y) d\mu(x) d\mu(y).$$

The *capacity with respect to* $F$ is

$$Cap_F(\Lambda) = [\inf F(\mu)]^{-1}$$

where the infimum is over probability measures $\mu$ on $(\Lambda, \mathcal{B})$ and by the convention, $\infty^{-1} = 0$.

**Theorem 6.** *Let $\{X_t, t > 0\}$ be a Markov process (as above) on $S$ and $\rho \in S$. For any closed transient subset $E$ of $S$ we have*

$$P_\rho(T_E < \infty) \leq Cap_K(E) \tag{7}$$

*where $K$ is the Martin kernel*

$$K(x,y) = \frac{g(x,y)}{g(\rho,y)}$$

*defined using the initial state $\rho$.*

**Proof.** To bound from above the probability of ever hitting $E$, consider the hitting time $T_E$ and the last exit time $\gamma_E$ of $E$, and the distribution

$$\nu_\rho(\Lambda) = L^E(\rho, \Lambda) = P_\rho(t < \gamma_E | x_{\gamma_E -} \in \Lambda); \ \Lambda \in \mathcal{B}(\mathbb{S})$$

The Kai Lai Chung's result says that

$$L^E(\rho, \Lambda) = \int_\Lambda g(x,y)\mu_E(dy); \ \Lambda \in \mathcal{B}(\mathbb{S})$$

where $\mu_E$ is the equilibrium measure of $E$, which is given by

$$\forall x \in \mathbb{S} : \mu_E(dy) = L^E(x, dy) g(x,y)^{-1} = \nu_x(dy) g(x,y)^{-1}$$

in particular, for the initial state $\rho \in \mathbb{S}$

$$\mu_E(dy) = L^E(\rho, dy) g(\rho, y)^{-1} = \nu_\rho(dy) g(\rho, y)^{-1}.$$

It follows that

$$\begin{aligned} \int_E K(x,y)\nu_\rho(dy) &= \int_E K(x,y) g(\rho,y) \mu_E(dy) \\ &= \int_E g(x,y)\mu_E(dy) = P_x(T_E < \infty) \leq 1. \end{aligned} \tag{8}$$

Therefore $K(\nu_\rho, \nu_\rho) \leq \nu_\rho(E)$ and thus

$$Cap_K(E) \geq [K(\nu_\rho / \nu_\rho(E))]^{-1} \geq \nu_\rho(E)$$

which by (8) yields the upper bound on the probability of hitting $E$. ■

Let $E$ a closed set of $\mathbb{S}$ and let

$$P(x, E) = P_x(Reach_\infty(E)).$$

For any $x \in \mathbb{S}$ we define the *harmonic measure* of $E$ as follows:

$$H^E(x,A) = P_x(X_{T_E} \in A), \ A \in \mathcal{B}(\mathbb{S}).$$

Clearly, the support of $H^E(x,\cdot)$ is $E$ and $P(x,E) = H^E(x,E)$.

An weaker result for the hitting probability estimation then the theorem 6 can be obtained starting with the following proposition:

**Proposition 7.** [23] *If the given process $\{X_t, t \geq 0\}$ has the strong Markov property then, for any transient closed set $E \in \mathcal{B}(\mathbb{S})$ and for $y \in E$ and $x \notin E$ the following equality holds*

$$g(x,y) = \int_E g(u,y) H^E(x,du).$$

The hitting probability estimation follows as:

**Corollary 8.** *Under the above hypotheses,*

$$P(x,E) \leq \frac{g(x,y)}{\inf_{u\in E} g(u,y)}, \ \forall y \in E$$

**Proof.** $g(x,y) = \int_E g(u,y)H^E(x,du) \leq \inf_{u\in E} g(u,y)H^E(x,E) = P(x,E)\inf_{u\in E} g(u,y)$. Thus

$$P(x,E) \leq \frac{g(x,y)}{\inf_{u\in E} g(u,y)}, \ \forall y \in E$$

■

## 5 Conclusions

The first part of this analysis considers the capacities associated to the product of two standard Markov processes with some nice properties while the second part focusses on a particular type of capacity, the Martin capacity, which forms a threshold for reach set probabilities.

## References

[1] Benjamini, I., Pemantle, R., Peres, Y., *Martin Capacity for Markov Chains.* Ann. Probab., 23, 1332-1346, 1995.

[2] Bertin, E. M. J., *Un Produit d'Espaces de Dirichlet*, Comm. Math. Rijk. Univ. Utrecht, **12**, 26-41, 1980.

[3] Blumenthal, R. M., Getoor, R. K., McKean, H. P., *Markov Processes with Identical Hitting Distributions. Illinois J.Math.* **6**, 402-420, 1962.

[4] Blumenthal, R. M., Getoor, R. K., McKean, H. P., *A supplement to "Markov Processes with Identical Hitting Distributions"*. Illinois J.Math. **7**, 540-542, 1963.

[5] Blumenthal, R. M., Getoor, R. K., *Markov Processes and Potential Theory*, Academic Press, New York and London, 1968.

[6] Boboc, N., Bucur, G., *Characterization of Dirichlet forms by their Excessive and Co-excessive Elements.* Potential Analysis **8**, 345-357, 1998.

[7] Borkar, V. S., *Probability Theory, An Advance Course.* Springer-Verlag New York, 1995.

[8] Bujorianu, M. L., Lygeros, J., *Reachability Questions in Piecewise Deterministic Markov Processes,* In O. Maler, A. Pnueli editori, Proceedings **Hybrid Systems: Computation and Control**, 6th International Workshop, HSCC 2003, Prague, Czech Republic, April 2003, Lecture Notes in Computer Science 2623, 126-140, 2003.

[9] Chung, K. L., *Probabilistic Approach in Potential Theory to the Equilibrium Problem.* Annales de L'Institut Fourier,Grenoble, 23, 313-322, 1973.

[10] Chung, K. L., *Green, Brown and Probability.* World Scientific Publishing Co. Pte. Ltd. 1995

[11] Dynkin, E. B., *Minimal Excessive Measures and Functions.* Trans. Amer. Math. Soc. 258, 217-244

[12] Ethier, S. N., Kurtz, T. G., *Markov Processes: Characterization and Convergence.* Wiley series in Probability and Statistics, John Wiley & Sons, 1986.

[13] Fitzsimmons, P. J., *Markov Processes with Identical Hitting Probabilities.* Math.Zeit. **192**, 547-554, 1986.

[14] Fitzsimmons, P. J., *On the Identification of Markov Processes by the Distribution of Hitting Times.* In Seminar on Stochastic Processes, Birkhäuser, Boston, 15-19, 1987.

[15] Fitzsimmons, P. J., *Markov Processes and non-Symmetric Dirichlet Forms without Regularity.* J. Funct. Anal. **85**, 287-306.

[16] Fitzsimmons, P. J., *Markov Processes with Equal Capacities.* J. Theor. Prob. **12**, 271-292, 1999.

[17] Fitzsimmons, P. J., Getoor, R. K., Sharpe, M. J., *The Blumenthal-Getoor-McKean theorem revisited.* In Seminar on Stochastic Processes, Birkhäuser, Boston, 35-57, 1989.

[18] Fukushima, M., *Dirichlet Forms and Markov Processes* North Holland, 1980.

[19] Getoor, R. K., Steffens, J., *The Energy Functional, Balayage, and Capacity.* Ann. Institut. H. Poincaré **23**, 321-357.

[20] Glover, J., *Markov Processes with Identical Hitting Probabilities.* Trans.Amer. Math. Soc. **275**, 131-141, 1983.

[21] Glover, J., *Identifying Markov Processes up to Time Change.* In Seminar on Stochastic Processes, Birkhäuser, Boston, 171-194, 1982.

[22] Glover, J., Hansen, W., Rao, M., *Capacities of Symmetric Markov Processes.* In Seminar on Stochastic Processes, Birkhäuser, Boston, 159-170, 1987.

[23] Kunita, H., Watanabe, T., *Markov Processes and Martin Boundaries*, Part I, Illinois J. Math. 9, 485-526, 1965.

[24] Ma, M., Rockner, M., *The Theory of (Non-Symmetric) Dirichlet Forms and Markov Processes,* Springer Verlag, Berlin, 1990.

[25] Meyer, P. A., *Processus de Markov.* Lecture Notes in Mathematics 26, Berlin, Heidelberg, New York: Springer 1967.

[26] Popa, E., *Tensor Product of Standard H-cones.* An. St. Univ. Iasi.