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Univalence Conditions for Integral Operators on $S(\alpha)$-Class

Daniel BREAZ, Nicoleta BREAZ

Abstract. We consider the class of univalent functions defined by the conditions $f(z)/z \neq 0$ and $|(z/f(z))''| \leq \alpha$, $|z| < 1$, where $f(z) = z + \ldots$ is analytic in $|z| < 1$ and $0 < \alpha \leq 2$. In this paper we prove two univalence conditions for the integral operators $H_{\beta,\gamma}$ and respectively G_γ.
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Introduction

Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc $U = \{z : |z| < 1\}$.

For $0 < \alpha \leq 2$, let $S(\alpha)$ denote the class of functions $f \in A$ which satisfy the conditions

$$f(z) \neq 0 \quad \text{for} \quad 0 < |z| < 1$$

and

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq \alpha, \quad z \in U. \tag{1}$$

If $\alpha = 2$ then the functions f which satisfy the property (1) are univalent and the bound 2 is the best.

For $0 < \alpha \leq 2$ the functions $f \in S(\alpha)$ are univalent.

If $f \in S(\alpha)$ then it satisfy the property

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq \alpha |z|^2 \tag{2}$$

for $z \in U$.

The property (2) can be found in [1] and is used in the next proofs.

Theorem A. [3] *Let* $\alpha \in C$, $\operatorname{Re}\alpha > 0$ *and* $f \in A$. *If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in U$$

then $\forall \beta \in C$, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, *the function*

$$F_{\beta}(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)\right]^{\frac{1}{\beta}}$$

is univalent.

Theorem A has been proved by N. N. Pascu and can be found in [3]. It is used for proving the results below.

The Schwarz Lemma. [4] *Let the analytic functions $f(z)$ be regular in the unit circle $|z| < 1$ and let $f(0) = 0$. If, in $|z| < 1$, $|f(z)| \leq 1$, then*

$$|f(z)| \leq |z|, \quad |z| < 1$$

where equality can hold only if $f(z) \equiv Kz$ and $|K| = 1$.

Main results

Theorem 1. *Let $f \in S(\alpha)$, $0 < \alpha \leq 2$, $f = z + a_3 z^3 + a_4 z^4 + \ldots$ and $\gamma \in C$ i. e.*

$$Re\gamma \geq \frac{\alpha+2}{|\gamma|}.$$

If $|f(z)| \leq 1$, $\forall\, z \in U$, then $\forall \beta \in C$, $\operatorname{Re}\beta \geq \operatorname{Re}\gamma$ the function

$$H_{\beta,\gamma}(z) = \left\{\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)^{\frac{1}{\gamma}} dt\right\}^{\frac{1}{\beta}} \in S$$

Proof. Let the function

$$h(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\frac{1}{\gamma}} dt.$$

We calculate the derivatives of order one and two of the function h.

$$h'(z) = \left(\frac{f(t)}{t}\right)^{\frac{1}{\gamma}}, \quad h''(z) = \frac{1}{\gamma}\left(\frac{f(t)}{t}\right)^{\frac{1}{\gamma}-1} \frac{zf'(z) - f(z)}{z^2}.$$

We have

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh''(z)}{h'(z)}\right| = \frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\frac{1}{|\gamma|}\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad \forall z \in U$$

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh''(z)}{h'(z)}\right| = \frac{1-|z|^{2\operatorname{Re}\gamma}}{|\gamma|\operatorname{Re}\gamma}\left|\frac{zf'(z)}{f(z)}\right| + \frac{1-|z|^{2\operatorname{Re}\gamma}}{|\gamma|\operatorname{Re}\gamma}, \quad \forall z \in U$$

Because

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh''(z)}{h'(z)}\right| = \frac{1-|z|^{2\operatorname{Re}\gamma}}{|\gamma|\operatorname{Re}\gamma}\left(\left|\frac{z^2 f'(z)}{f^2(z)}\right|\frac{|f(z)|}{|z|} + 1\right), \quad \forall z \in U$$

Using the hypothesis of the theorem from Scwartz's Lemma we obtain

$$|f(z)| \leq |z|, \quad \forall z \in U. \tag{4}$$

Using equation (4) we obtain

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh''(z)}{h'(z)}\right| = \frac{1-|z|^{2\operatorname{Re}\gamma}}{|\gamma|\operatorname{Re}\gamma}\left(\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| + 2\right), \quad \forall z \in U.$$

But

$$f \in S(\alpha) \Rightarrow \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq \alpha |z|^2, \quad \forall z \in U.$$

Going back to the proof of the theorem from the last relation we obtain

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh''(z)}{h'(z)}\right| = \frac{1-|z|^{2\operatorname{Re}\gamma}}{|\gamma|\operatorname{Re}\gamma}\left(\alpha|z|^2+2\right) \leq \frac{\alpha+2}{|\gamma|\operatorname{Re}\gamma}\left(1-|z|^{2\operatorname{Re}\gamma}\right) \leq \frac{\alpha+2}{|\gamma|\operatorname{Re}\gamma}, \forall z \in U$$

Since according to the hypothesis

$$Re\gamma \geq \frac{\alpha+2}{|\gamma|}$$

we have

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad \forall z \in U,$$

Thus, from Theorem A we obtain $H_{\beta,\gamma} \in S$

Theorem 2. *Let $f \in S(\alpha)$, $0 < \alpha \leq 2$, $f = z + a_3 z^3 + a_4 z^4 + \ldots$ and $\gamma \in C$ i. e.*

$$|\gamma - 1| \leq \frac{\operatorname{Re}\gamma}{\alpha+2}$$

If $|f(z)| \leq 1$, $\forall\ z \in U$, then the function

$$G_\gamma(z) = \left\{\gamma \int_0^z f^{\gamma-1}(t)\,dt\right\}^{\frac{1}{\gamma}} \in S$$

Proof. The function G_γ can be written as

$$G_\gamma(z) = \left\{\gamma \int_0^z t^{\gamma-1}\left(\frac{f(t)}{t}\right)^{\gamma-1} dt\right\}^{\frac{1}{\gamma}}$$

We consider the function $g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\gamma-1} dt$

The first and second order derivative of the function g are

$$g'(z) = \left(\frac{f(t)}{t}\right)^{\frac{1}{\gamma}}, \quad g''(z) = (\gamma-1)\left(\frac{f(t)}{t}\right)^{\gamma-2}\frac{zf'(z)-f(z)}{z^2}.$$

We have

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}|\gamma-1|\left(\left|\frac{zf'(z)}{f(z)}\right|+1\right), \quad \forall z \in U$$

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq |\gamma-1|\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left(\left|\frac{z^2 f'(z)}{f^2(z)}\right|\frac{|f(z)|}{|z|}+1\right) \leq$$

$$\leq |\gamma-1|\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left(\left|\frac{z^2 f'(z)}{f^2(z)}-1\right|+2\right), \quad \forall z \in U$$

Since $f \in S(\alpha) \Rightarrow \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq \alpha|z|^2, \forall z \in U.$

Using the above relation we obtain

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \le \frac{|\gamma-1|}{\operatorname{Re}\gamma}\left(1-|z|^{2\operatorname{Re}\gamma}\right)\left(\alpha|z|^2+2\right) \le$$
$$\le \frac{|\gamma-1|}{\operatorname{Re}\gamma}\left(1-|z|^{2\operatorname{Re}\gamma}\right)(\alpha+2) \le |\gamma-1|\frac{\alpha+2}{\operatorname{Re}\gamma}, \quad \forall z\in U$$

Using the hypothesis

$$|\gamma-1| \le \frac{\operatorname{Re}\gamma}{\alpha+2},$$

we obtain

$$\frac{1-|z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \le 1, \quad \forall z\in U$$

and from Theorem A we have $G_\gamma \in S$.

Particular cases

An interesting particular case of Theorem 1 is obtain if we take γ to be equal to 2. If this is the case we obtain the following result

Corollary 1. *Let $f \in S(\alpha)$, $0<\alpha\le 2$, $f = z + a_3z^3 + a_4z^4 + \dots$. If $|f(z)| \le 1$, $\forall z \in U$, then $\forall \beta \in C$, $\operatorname{Re}\beta \ge 2$ the function*

$$H_\beta(z) = \left\{\beta\int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\frac{1}{2}} dt\right\}^{\frac{1}{\beta}} \in S$$

Remark 2. We note that the condition $Re\gamma \ge \frac{\alpha+2}{|\gamma|}$ may be eliminated although it is essential for Theorem 1.

If we take β to be equal to 2, such that $\operatorname{Re}\beta \ge 2$ then we obtain the following result

Corollary 3. *Let $f \in S(\alpha)$, $0<\alpha\le 2$, $f = z + a_3z^3 + a_4z^4 + \dots$. If $|f(z)| \le 1$, $\forall z \in U$, then the function*

$$H(z) = \left\{2\int_0^z \sqrt{tf(t)}dt\right\}^{\frac{1}{2}} \in S$$

References

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