

Classification of Quadratic Differential Systems on \mathbb{R}^3 having a Nilpotent of Order 3 Derivation

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Abstract. This paper deals with the classification, up to a center-affinity, of the quadratic differential systems defined on \mathbb{R}^3 which have a nilpotent of order 3 derivation. It is proved that there exist nine classes of nonequivalent such systems.

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1 Introduction

The study of homogeneous quadratic differential system S (HQDS) on a BANACH space A can be achieved algebraically because a commutative algebra $A(\cdot)$ is naturally associated with any such a system. For convenience, we say that a derivation/automorphism of $A(\cdot)$ is just a derivation/automorphism of S .

One of the most important problem in the theory of HQDSs on a real finite dimensional vector space (e.g., \mathbb{R}^n) is their classification up to a center-affine equivalence. This problem is equivalent with the problem of classification up to an isomorphism of the real finite dimensional commutative algebras. Such classifications were already obtained for HQDSs on \mathbb{R}^2 (see [6], [4], [9], etc.) in accordance with different classifying criteria. Further, particular classes of HQDSs on \mathbb{R}^3 were classified by KINYON&SAGLE [5]. Other classifying results, up to an affine equivalence, were obtained for the real systems on \mathbb{R}^3 whose associated algebras have no nilpotent of order two elements but have at least a derivation ([2]) as well as for the real systems which have at least a derivation with a complex eigenvalue ([3]). In this paper, the HQDSs which have a nilpotent of order three derivation are classified. More exactly, it was proved that there exist nine classes of affinely nonequivalent such HQDSs. Our results are founded on the use of invariant objects such as the annihilators, nilpotents, idempotents, derivation algebras and automorphism groups.

2 Preliminaries

Recall that a *homogeneous quadratic differential system* (briefly, HQDS) on \mathbb{R}^n is any autonomous system of the form

$$\frac{dx^i}{dt} = a_{jk}^i x^j x^k \quad (2.1)$$

where $a_{jk}^i \in \mathbb{R}$, $a_{jk}^i = a_{kj}^i$ (here it was used EINSTEIN'S summing convention).

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible linear transformation, $B = (e_1, e_2, \dots, e_n)$ - a fixed basis in \mathbb{R}^n and $[T]_B = [t_j^i]$ be the matrix of T in basis B . If $y^i = t_j^i x^j$ are the equations of T , then (2.1) becomes

$$\frac{dy^i}{dt} = \tilde{a}_{jk}^i y^j y^k, \quad \tilde{a}_{jk}^i = t_s^i a_{pq}^s \tilde{t}_j^p \tilde{t}_k^q, \quad (2.2)$$

where $[\tilde{t}_j^i]$ denotes the inverse matrix for $[t_j^i]$. Second part of (2.2) assures us that \tilde{a}_{jk}^i are, alike the structure constants of any finite dimensional algebra, the components of an (1,2)-tensor in a given basis.

Definition 2.1 *Two HQDSs are called (center-)affinely equivalent if there exists an invertible linear transformation of \mathbb{R}^n which transforms one HQDS in the other.*

Consequently, an (1,2)-tensor is naturally defined by the class of all HQDSs which are affinely equivalent with a given HQDS. L. MARKUS remarked that such a tensor is also uniquely defined by the class of all commutative algebras isomorphic with a given algebra. From here arisen the idea to associate with any HQDS a commutative algebra. More exactly, the binary commutative algebra on \mathbb{R}^n which has in the chosen basis the structure constants \tilde{a}_{jk}^i is associated with the HQDS (2.1)(see [6], [1]). It was proved [1] that *the problem of classification up to an affine equivalence of HQDSs on \mathbb{R}^n is equivalent with the problem of classification up to an isomorphism of all n -dimensional real commutative algebras.*

3 The classification of algebras which have a nilpotent of order 3 derivation

Let $A(\cdot)$ be the real 3-dimensional commutative algebra associated with a HQDS in \mathbb{R}^3 ; obviously, A can be naturally identified with \mathbb{R}^3 .

A derivation $\tilde{D} \neq 0$ of $A(\cdot)$ is said to be *nilpotent of order 3* if $\tilde{D}^3 = 0$ and $\tilde{D}^2 \neq 0$. Therefore, its JORDAN form consists in a single JORDAN cell, i.e. $\dim \ker \tilde{D} = \text{codim } \text{Im } \tilde{D} = 1$ and $\ker \tilde{D} \subset \text{Im } \tilde{D}$. Accordingly, there exists a basis $\mathcal{B} = (e_1, e_2, e_3)$ for A in which the matrix of \tilde{D} has the form:

$$[\tilde{D}]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

i.e., the following equalities hold

$$\tilde{D}e_1 = 0, \quad \tilde{D}e_2 = e_1, \quad \tilde{D}e_3 = e_2.$$

Recall that the structure constants of the algebra in basis \mathcal{B} are defined by: $e_j \cdot e_k = a_{jk}^i e_i$.

Applying successively \tilde{D} to $e_j \cdot e_k$ one obtains the relations:

$$e_1^2 = a_{11}^i e_i \xrightarrow{\tilde{D}} e_1^2 = a_{11}^1 e_1,$$

$$e_1 \cdot e_2 = a_{12}^i e_i \xrightarrow{\tilde{D}} e_1^2 = a_{12}^2 e_1 + a_{12}^3 e_2 = a_{11}^1 e_1 \Rightarrow e_1 \cdot e_2 = a_{12}^1 e_1 + a_{11}^1 e_2,$$

$$e_1 \cdot e_3 = a_{13}^i e_i \xrightarrow{\tilde{D}} e_1 \cdot e_2 = a_{13}^2 e_1 + a_{13}^3 e_2 \Rightarrow e_1 \cdot e_3 = a_{13}^1 e_1 + a_{12}^1 e_2 + a_{11}^1 e_3,$$

$$e_2^2 = a_{22}^i e_i \xrightarrow{\tilde{D}} 2e_1 \cdot e_2 = a_{22}^2 e_1 + a_{22}^3 e_2 \Rightarrow e_2^2 = a_{22}^1 e_1 + 2a_{12}^1 e_2 + 2a_{11}^1 e_3,$$

$$e_2 \cdot e_3 = a_{23}^i e_i \xrightarrow{\tilde{D}} e_1 \cdot e_3 + e_2^2 = a_{23}^2 e_1 + a_{23}^3 e_2 \Rightarrow a_{11}^1 = 0, \Rightarrow e_1^2 = 0,$$

$$e_2 \cdot e_3 = a_{23}^1 e_1 + (a_{13}^1 + a_{22}^1) e_2 + 3a_{12}^1 e_3, \quad e_1 \cdot e_2 = a_{12}^1 e_1, \quad e_1 \cdot e_3 = a_{13}^1 e_1 + a_{12}^1 e_2,$$

$$e_2^2 = a_{22}^1 e_1 + 2a_{12}^1 e_2$$

$$e_3^2 = a_{33}^i e_i \xrightarrow{\tilde{D}} 2e_2 \cdot e_3 = a_{33}^2 e_1 + a_{33}^3 e_2 \Rightarrow a_{12}^1 = 0 \Rightarrow e_1 \cdot e_2 = 0, \quad e_2^2 = a_{22}^1 e_1,$$

$$e_1 \cdot e_3 = a_{13}^1 e_1,$$

$$e_2 \cdot e_3 = a_{23}^1 e_1 + (a_{13}^1 + a_{22}^1) e_2.$$

Consequently, in basis \mathcal{B} , algebra $A(\cdot)$ has the following multiplication table

Table T	$e_1^2 = 0,$	$e_2^2 = a_{22}^1 e_1,$
	$e_1 e_2 = 0,$	$e_2 e_3 = a_{23}^1 e_1 + (a_{13}^1 + a_{22}^1) e_2,$
	$e_1 e_3 = a_{13}^1 e_1,$	$e_3^2 = a_{33}^1 e_1 + 2a_{23}^1 e_2 + 2(a_{13}^1 + a_{22}^1) e_3.$

The structure of algebra A . We point out that $\ker \tilde{D} = \mathbb{R}e_1$ and $\text{Im } \tilde{D} = \text{Span}_{\mathbb{R}} \{e_1, e_2\}$ are nilpotent ideals of algebra $A(\cdot)$. Moreover, $\text{Im } \tilde{D}$ is just the maximal nilpotent/solvable ideal.

If $a_{13}^1 + a_{22}^1 \neq 0$ then $A^2 = A$ and there exists the sequence of ideals

$$\{0\} \subset \ker \tilde{D} \subset \text{Im } \tilde{D} \subset A, \quad (3.1)$$

while, when $a_{13}^1 + a_{22}^1 = 0$ then $A^2 = \text{Span}_{\mathbb{R}} \{e_1, e_2\}$ and $A(\cdot)$ is a solvable algebra having the derived series

$$\{0\} \subset \ker \tilde{D} \subset \text{Im } \tilde{D} = A^2 \subset A. \quad (3.2)$$

Consequently, the vanishing/nonvanishing of $a_{13}^1 + a_{22}^1$ has an invariant character (i.e., it do not depend on the adapted basis which is used). Further, $\text{Im } \tilde{D}$ is or not a trivial algebra, in accordance with a_{22}^1 is zero or is different of zero. Therefore, the vanishing/nonvanishing of a_{22}^1 has an invariant character, too.

The existence of the sequences (3.1) and (3.2) creates the opportunity to consider some special basis builded in the following way: *take a basis in $\ker \tilde{D}$ and complete it up to a basis in $\text{Im } D$, which - in its turn - be complete up to a basis in A .* Any such a basis is named

an *adapted basis to the sequence* (3.1) or (3.2) or, briefly, an *adapted basis*. Obviously, any adapted basis $\mathcal{B}' = (f_1, f_2, f_3)$ for A has necessarily the form

$$f_1 = \alpha e_1, \quad f_2 = \beta e_1 + \gamma e_2, \quad f_3 = \delta e_1 + \varepsilon e_2 + \theta e_3,$$

with $\alpha\gamma\theta \neq 0$. Actually, the family of all adapted bases is very large. It is suffice to remark that for any vector of the adapted basis there exist infinitely many possibilities for its choice.

Remark 3.1 *As it was already remarked, the vanishing/nonvanishing of $a_{13}^1 + a_{22}^1$ is reflected in the structure of algebra, so that it has an invariant character. This invariance results also from the remark that L_{e_3} has the eigenvalues $\lambda_1 = a_{13}^1$, $\lambda_2 = a_{13}^1 + a_{22}^1$ and $\lambda_3 = 2(a_{13}^1 + a_{22}^1) = 2\lambda_2$. Moreover, the third component of any adapted basis have the eigenvalues of its left associated multiplications proportional with $\lambda_1, \lambda_2, \lambda_3 = 2\lambda_2$. Since a_{13}^1 is also an eigenvalue for L_{e_3} it results, by the same argument, that its vanishing/nonvanishing has an invariant character, too.*

Remark 3.2 *The adapted basis permits us to identify the multiplication algebra $\mathcal{M}(A)$ (see [8], p.14) with a subalgebra of the algebra of upper triangle 3×3 -matrices for which the eigenvalues are proportional with $\lambda_1, \lambda_2, \lambda_3 = 2\lambda_2$. Indeed, it is enough to notice that for $v = xe_1 + ye_2 + ze_3$ its left multiplication L_v has (in an adapted basis) the matrix*

$$[L_v]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 z & a_{22}^1 y + a_{23}^1 z & a_{13}^1 x + a_{23}^1 y + a_{33}^1 z \\ 0 & \lambda_2 z & (a_{13}^1 + a_{22}^1) y + 2a_{23}^1 z \\ 0 & 0 & \lambda_3 z \end{bmatrix}.$$

The annihilators of algebra A . Let determine the ideal $\text{Ann } A$ of all annihilators of algebra A . Recall that a nonzero element $v \in A$ is called its *annulator* if $vw = 0$, for all $w \in A$, i.e. left multiplication L_v is just the null endomorphism of the vector space A . Therefore, in any basis \mathcal{B} , $[L_v]_{\mathcal{B}} = 0$. It results that $v = xe_1 + ye_2 + ze_3$ is an annulator if and only if

$$\begin{cases} a_{13}^1 z = 0, \\ a_{22}^1 y + a_{23}^1 z = 0, \\ (a_{13}^1 + a_{22}^1) z = 0, \\ a_{13}^1 x + a_{23}^1 y + a_{33}^1 z = 0, \\ (a_{13}^1 + a_{22}^1) y + 2a_{23}^1 z = 0. \end{cases}$$

This system allows to prove the following result.

Proposition 3.1 *Algebra $A(\cdot)$ has at least an annulator element if and only if $a_{13}^1 = 0$. Moreover, it results*

- 1° $\text{Ann } A = \{0\}$ iff $a_{13}^1 \neq 0$,
- 2° $\text{Ann } A = \mathbb{R}e_1$ iff $a_{13}^1 = 0$ and $(a_{22}^1)^2 + (a_{23}^1)^2 \neq 0$,
- 3° $\text{Ann } A = \text{Span}_{\mathbb{R}}\{e_1, e_2\}$ iff $a_{13}^1 = a_{22}^1 = a_{23}^1 = 0$ and $a_{33}^1 \neq 0$,
- 4° $\text{Ann } A = A$ iff $a_{13}^1 = a_{22}^1 = a_{23}^1 = a_{33}^1 = 0$.

The nilpotents of algebra A . Recall that a nonzero element $v \in A$ is named a *nilpotent* of A if $v^2 = 0$. It is suitable to accept that $0 \in A$ is always a nilpotent. The set of all nilpotents of algebra A is denoted by $\mathcal{N}(A)$.

For any $v = xe_1 + ye_2 + ze_3 \in A$, the condition $v^2 = 0$ is equivalent with the quadratic algebraic system

$$\begin{cases} a_{22}^1 y^2 + a_{33}^1 z^2 + 2a_{13}^1 xz + 2a_{23}^1 yz = 0 \\ 2a_{23}^1 z^2 + 2(a_{13}^1 + a_{22}^1)yz = 0 \\ 2(a_{13}^1 + a_{22}^1)z^2 = 0, \end{cases}$$

Analyzing this system the following result was proved.

Proposition 3.2 *Algebra $A(\cdot)$ has the nilpotents:*

a) $\mathcal{N}(A) = \mathbb{R}e_1$ if it is fulfilled one of the conditions:

- 1) $a_{13}^1 + a_{22}^1 \neq 0$ and $a_{22}^1 \neq 0$,
- 2) $a_{13}^1 + a_{22}^1 = 0$ and $a_{23}^1 \neq 0$, $a_{22}^1 \neq 0$

b) $\mathcal{N}(A) = \text{Span}_{\mathbb{R}}\{e_1, e_2\}$ if it is fulfilled one of the conditions:

- 1) $a_{13}^1 + a_{22}^1 \neq 0$ and $a_{22}^1 = 0$,
- 2) $a_{13}^1 = a_{22}^1 = 0$ and $a_{23}^1 \neq 0$,
- 3) $a_{13}^1 = a_{22}^1 = a_{23}^1 = 0$ and $a_{33}^1 \neq 0$,

c) $\mathcal{N}(A) = \left\{ \frac{-1}{2a_{22}^1 z} (a_{22}^1 y^2 + a_{33}^1 z^2) e_1 + ye_2 + ze_3 \mid \forall y \in \mathbb{R}, z \in \mathbb{R}^* \right\}$ if $a_{13}^1 + a_{22}^1 = 0$ and $a_{23}^1 = 0$ and $a_{22}^1 \neq 0$

d) $\mathcal{N}(A) = A$ if $a_{13}^1 = a_{22}^1 = a_{23}^1 = a_{33}^1 = 0$.

The idempotents of algebra A . In what follows we denote by $\mathcal{I}(A)$ the set of all idempotents of algebra $A(\cdot)$.

The condition $v^2 = v$ is equivalent with:

$$\begin{cases} a_{22}^1 y^2 + a_{33}^1 z^2 + 2a_{13}^1 xz + 2a_{23}^1 yz = x \\ 2a_{23}^1 z^2 + 2(a_{13}^1 + a_{22}^1)yz = y \\ 2(a_{13}^1 + a_{22}^1)z^2 = z, \end{cases}$$

It can be proved the following result.

Proposition 3.3 *Algebra $A(\cdot)$ has the properties:*

1) if $a_{13}^1 + a_{22}^1 \neq 0$, $a_{23}^1 = 0$ and $a_{22}^1 \neq 0$ then

$$\mathcal{I}(A) = \left\{ \frac{1}{a_{22}^1} \frac{4(a_{13}^1 + a_{22}^1)^2 a_{22}^1 y^2 + a_{33}^1}{4(a_{13}^1 + a_{22}^1)} e_1 + ye_2 + \frac{1}{2(a_{13}^1 + a_{22}^1)} e_3 \mid \forall y \in \mathbb{R} \right\},$$

2) if $a_{13}^1 + a_{22}^1 \neq 0$, $a_{23}^1 = 0$, $a_{22}^1 = 0$, $a_{33}^1 = 0$, then

$$\mathcal{I}(A) = \left\{ xe_1 + ye_2 + \frac{1}{2a_{13}^1} e_3 \mid \forall x, y \in \mathbb{R} \right\},$$

2) in the other cases do not exist idempotents, i.e., $\mathcal{I}(A) = \emptyset$.

Canonical bases for A . We look for a basis $B' = (f_1, f_2, f_3) \subset A$ such that the corresponding structure constants of the algebra be as simple as possible (i.e., a maximal number of them become 0 or an integer). Obviously, we shall use adapted bases to the before exhibited composition sequences, what solve a part of the problem. Therefore, taking into account of the existence of sequence (3.1), we look for a basis \mathcal{B}' of the form:

$$\begin{cases} f_1 = \alpha e_1 \\ f_2 = \beta e_1 + \gamma e_2 \\ f_3 = \delta e_1 + \varepsilon e_2 + \theta e_3, \end{cases}$$

with $\alpha\gamma\theta \neq 0$. Then

$$\begin{aligned} f_1^2 &= f_1 \cdot f_2 = 0, \\ f_1 \cdot f_3 &= \theta a_{13}^1 f_1, \quad f_2^2 = \frac{\gamma^2}{\alpha} a_{22}^1 f_1, \\ f_2 \cdot f_3 &= \frac{1}{\alpha} [\gamma(\varepsilon a_{22}^1 + \theta a_{23}^1) - \beta \theta a_{22}^1] f_1 + \theta(a_{13}^1 + a_{22}^1) f_2, \\ f_3^2 &= \frac{1}{\alpha} \left\{ \varepsilon^2 a_{22}^1 + \theta^2 a_{33}^1 + 2\varepsilon \theta a_{23}^1 - 2\frac{\beta\theta^2}{\gamma} a_{23}^1 - 2\delta\theta a_{22}^1 \right\} f_1 + \\ &\quad + 2\frac{\theta^2}{\gamma} a_{23}^1 f_2 + 2\theta(a_{13}^1 + a_{22}^1) f_3 \end{aligned}$$

The parameters $\alpha, \beta, \gamma, \delta, \varepsilon, \theta$ will be determined such that a maximal number of coefficients vanish or take integer values. The corresponding basis will be called a *canonical basis*. Even if it exists, a canonical basis is not necessarily a unique one. Actually, the family of all canonical basis is closely connected with the group of automorphisms of the algebra.

One distinguishes the following 9 situations:

$$\begin{aligned} T_1 : & a_{13}^1 \neq 0, \quad a_{22}^1 \neq 0, \quad a_{23}^1 \neq 0, \\ T_2 : & a_{13}^1 \neq 0, \quad a_{22}^1 \neq 0, \quad a_{23}^1 = 0, \\ T_3 : & a_{13}^1 \neq 0, \quad a_{22}^1 = 0, \quad a_{23}^1 \neq 0, \\ T_4 : & a_{13}^1 \neq 0, \quad a_{22}^1 = 0, \quad a_{23}^1 = 0, \quad a_{33}^1 \neq 0, \\ T_5 : & a_{13}^1 \neq 0, \quad a_{22}^1 = 0, \quad a_{23}^1 = 0, \quad a_{33}^1 = 0, \\ T_6 : & a_{13}^1 = 0, \quad a_{22}^1 \neq 0, \quad a_{23}^1 \neq 0, \\ T_7 : & a_{13}^1 = 0, \quad a_{22}^1 \neq 0, \quad a_{23}^1 = 0, \\ T_8 : & a_{13}^1 = 0, \quad a_{22}^1 = 0, \quad a_{23}^1 \neq 0, \\ T_9 : & a_{13}^1 = 0, \quad a_{22}^1 = 0, \quad a_{23}^1 = 0, \quad a_{33}^1 \neq 0. \end{aligned}$$

Case T_1 : $a_{13}^1 \neq 0, \quad a_{22}^1 \neq 0, \quad a_{23}^1 \neq 0$

Choosing

$$\begin{aligned} \theta &= \frac{1}{a_{13}^1}, \quad \gamma = 2\theta^2 a_{23}^1, \quad \alpha = \gamma^2 a_{22}^1, \quad \beta = \frac{\gamma}{\theta a_{22}^1} (\varepsilon a_{22}^1 + \theta a_{23}^1), \\ \delta &= \frac{1}{2\theta a_{22}^1} \left(\varepsilon^2 a_{22}^1 + \theta^2 a_{33}^1 + 2\varepsilon \theta a_{23}^1 - 2\frac{\beta\theta^2}{\gamma} a_{23}^1 \right), \end{aligned}$$

(ε can take any allowed value, eventually $\varepsilon = 0$) and putting $a = 1 + \frac{a_{22}^1}{a_{13}^1}$ the multiplication table of the algebra becomes:

$$\textbf{Table 1} \quad \begin{array}{lll} f_1^2 = 0 & f_1 f_2 = 0 & f_1 f_3 = f_1 \\ f_2^2 = f_1 & f_2 f_3 = a f_2 & f_3^2 = f_2 + 2a f_3 \end{array}$$

with $a \in \mathbb{R} \setminus \{1\}$.

Therefore it was obtained a uniparametric family of such algebras.

Case T_2 : $a_{13}^1 \neq 0$, $a_{22}^1 \neq 0$, $a_{23}^1 = 0$

Choosing

$$\theta = \frac{1}{a_{13}^1}, \quad \alpha = \gamma^2 a_{22}^1, \quad \beta = \frac{\gamma}{\theta a_{22}^1} (\varepsilon a_{22}^1 + \theta a_{23}^1), \quad \delta = \frac{1}{2\theta a_{22}^1} (\varepsilon^2 a_{22}^1 + \theta^2 a_{33}^1),$$

(γ and ε can take any allowed value, eventually, $\gamma = 1$ and $\varepsilon = 0$) and putting $a = 1 + \frac{a_{22}^1}{a_{13}^1}$ the multiplication table of the algebra becomes:

$$\textbf{Table 2} \quad \begin{array}{lll} f_1^2 = 0 & f_1 f_2 = 0 & f_1 f_3 = f_1 \\ f_2^2 = f_1 & f_2 f_3 = a f_2 & f_3^2 = 2a f_3 \end{array}$$

with $a \in \mathbb{R} \setminus \{1\}$.

It was again obtained a uniparametric family of algebras.

Case T_3 : $a_{13}^1 \neq 0$, $a_{22}^1 = 0$, $a_{23}^1 \neq 0$

Choosing

$$\theta = \frac{1}{a_{13}^1}, \quad \alpha = \gamma \theta a_{23}^1, \quad \varepsilon = \frac{\theta}{2\gamma a_{23}^1} (2\beta a_{23}^1 - \gamma a_{33}^1), \quad \gamma = 2\theta^2 a_{23}^1,$$

(β and δ can take any allowed value, eventually, $\beta = \delta = 0$) the multiplication table of the algebra becomes:

$$\textbf{Table 3} \quad \begin{array}{lll} f_1^2 = 0 & f_1 f_2 = 0 & f_1 f_3 = f_1 \\ f_2^2 = 0 & f_2 f_3 = f_1 + f_2 & f_3^2 = f_2 + 2f_3 \end{array}$$

Case T_4 : $a_{13}^1 \neq 0$, $a_{22}^1 = 0$, $a_{23}^1 = 0$, $a_{33}^1 \neq 0$

Choosing

$$\theta = \frac{1}{a_{13}^1}, \quad \alpha = \theta^2 a_{33}^1,$$

($\beta, \gamma, \delta, \varepsilon$ can take any allowed value, eventually, $\beta = \delta = \varepsilon = 0, \gamma = 1$) the multiplication table of the algebra becomes:

$$\textbf{Table 4} \quad \begin{array}{lll} f_1^2 = 0 & f_1 f_2 = 0 & f_1 f_3 = f_1 \\ f_2^2 = 0 & f_2 f_3 = f_2 & f_3^2 = f_1 + 2f_3 \end{array}$$

Case T_5 : $a_{13}^1 \neq 0$, $a_{22}^1 = 0$, $a_{23}^1 = 0$, $a_{33}^1 = 0$

Choosing $\theta = \frac{1}{a_{13}}$, ($\alpha, \beta, \gamma, \delta, \varepsilon$ can take any allowed value, eventually, $\beta = \delta = \varepsilon = 0$, $\alpha = \gamma = 1$) the multiplication table of the algebra becomes:

$$\textbf{Table 5} \quad \begin{array}{lll} f_1^2 = 0 & f_1 f_2 = 0 & f_1 f_3 = f_1 \\ f_2^2 = 0 & f_2 f_3 = f_2 & f_3^2 = 2f_3 \end{array}$$

Case T_6 : $a_{13}^1 = 0$, $a_{22}^1 \neq 0$, $a_{23}^1 \neq 0$

Choosing

$$\begin{aligned} \alpha &= \gamma^2 a_{22}^1, \quad \theta = \frac{1}{a_{22}^1}, \quad \beta = \gamma(\varepsilon a_{22}^1 + \theta a_{23}^1), \quad \gamma = 2\theta^2 a_{23}^1, \\ \delta &= \frac{1}{2\gamma}[\gamma(\varepsilon^2 a_{22}^1 + \theta^2 a_{33}^1 + 2\varepsilon\theta a_{23}^1) - 2\beta\theta^2 a_{23}^1], \end{aligned}$$

(ε can take any allowed value, eventually, $\varepsilon = 0$) the multiplication table of the algebra becomes:

$$\textbf{Table 6} \quad \begin{array}{lll} f_1^2 = 0 & f_1 f_2 = 0 & f_1 f_3 = 0 \\ f_2^2 = f_1 & f_2 f_3 = f_2 & f_3^2 = f_2 + 2f_3 \end{array}$$

Case T_7 : $a_{13}^1 = 0$, $a_{22}^1 \neq 0$, $a_{23}^1 = 0$

Choosing

$$\alpha = \gamma^2 a_{22}^1, \quad \theta = \frac{1}{a_{22}^1}, \quad \beta = \gamma\varepsilon a_{22}^1, \quad \delta = \frac{1}{2}(\varepsilon^2 a_{22}^1 + \theta^2 a_{33}^1)$$

(γ and ε can take any allowed value, eventually, $\gamma = 1$ and $\varepsilon = 0$) the multiplication table of the algebra becomes:

$$\textbf{Table 7} \quad \begin{array}{lll} f_1^2 = 0 & f_1 f_2 = 0 & f_1 f_3 = 0 \\ f_2^2 = f_1 & f_2 f_3 = f_2 & f_3^2 = 2f_3 \end{array}$$

Case T_8 : $a_{13}^1 = 0$, $a_{22}^1 \neq 0$, $a_{23}^1 = 0$

Choosing

$$\alpha = \gamma\theta a_{23}^1, \quad \gamma = 2\theta^2 a_{23}^1, \quad \beta = \frac{1}{2}(\theta^2 a_{33}^1 + 2\varepsilon\theta a_{23}^1)$$

(ε, δ and θ can take any allowed value, eventually, $\theta = 1$ and $\varepsilon = \delta = 0$) the multiplication table of the algebra becomes:

$$\textbf{Table 8} \quad \begin{array}{lll} f_1^2 = 0 & f_1 f_2 = 0 & f_1 f_3 = 0 \\ f_2^2 = 0 & f_2 f_3 = f_1 & f_3^2 = f_2 \end{array}$$

Case T_9 : $a_{13}^1 = 0$, $a_{22}^1 = 0$, $a_{23}^1 = 0$, $a_{33}^1 \neq 0$.

Choosing

$$\alpha = \theta^2 a_{33}^1$$

($\beta, \gamma, \delta, \varepsilon$ and θ can take any allowed value, eventually, $\gamma = \theta = 1$ and $\beta = \delta = \varepsilon = 0$) the multiplication table of the algebra becomes:

$$\textbf{Table 9} \quad \begin{array}{lll} f_1^2 = 0 & f_1 f_2 = 0 & f_1 f_3 = 0 \\ f_2^2 = 0 & f_2 f_3 = 0 & f_3^2 = f_1 \end{array}$$

Using the Propositions 3.1, 3.2 and 3.3 as well as some straightforward computations it can be established the main properties of algebras T1–T9.

Properties of algebras **Ti** ($i=1, \dots, 9$)

In what follows we shall denote by $A(Ti)$ or $A(Ti, a)$ any algebra having the multiplication table **Ti** corresponding to the parameter a if it is the case; for brevity, we put A instead of $A(Ti)$ or $A(Ti, a)$ when no confusion is possible.

Algebra **T1**

- $Ann(A) = \{0\}$, $\mathcal{N}(A) = \mathbb{R}f_1$, $\mathcal{I}(A) = \emptyset$,
- algebras $A(T1, a_1)$ and $A(T1, a_2)$ with $a_1 \neq 1$, $a_2 \neq 1$ are isomorphic if and only if $a_1 = a_2$,
- $A^2 = A$ if $a \neq 0$ and $A^2 = Im \tilde{D}$ if $a = 0$,
- $Der A = \mathbb{R}\tilde{D}$,
- $Aut A = \{T(\alpha) \mid \forall \alpha \in \mathbb{R}\}$ where

$$T(\alpha) = \begin{bmatrix} 1 & \alpha & \frac{b^2\alpha^2 - \alpha}{2b} \\ 0 & 1 & b\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

with $\alpha \in \mathbb{R}$ and $b = a - 1 \neq 0$.

Algebra **T2**

- $Ann(A) = \{0\}$, $\mathcal{N}(A) = \mathbb{R}f_1$ if $a \neq 0$ and $\mathcal{N}(A) = \mathbb{R}f_1 \cup \left\{ -\frac{y^2}{2z}f_1 + yf_2 + zf_3 \mid \forall y \in \mathbb{R}, z \in \mathbb{R}^* \right\}$ if $a = 0$,
- $\mathcal{I}(A) = \left\{ \frac{ay^2}{a-1}f_1 + yf_2 + \frac{1}{2a}f_3 \mid \forall y \in \mathbb{R} \right\}$, if $a \neq 0$ and $\mathcal{I}(A) = \emptyset$ if $a = 0$,
- algebras $A(T2, a_1)$ and $A(T2, a_2)$ with $a_1 \neq 1$, $a_2 \neq 1$ are isomorphic if and only if $a_1 = a_2$,
- $A^2 = A$ if $a \neq 0$ and $A^2 = Ker \tilde{D}$ if $a = 0$,
- $Der A = \mathbb{R}D_1 \oplus \mathbb{R}D_2$ where $[D_1]_{B'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$, $[D_2]_{B'} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
- $[D_1, D_2] = -D_1$ and $\tilde{D} = \frac{\gamma}{\alpha}D_1$,

- $Aut A = \{T(\alpha, \beta) \mid \forall \alpha \in \mathbb{R}\}$ where

$$T(\alpha, \beta) = \begin{bmatrix} \alpha^2 & \frac{1}{b}\alpha\beta & \frac{1}{2b}\beta^2 \\ 0 & \alpha & b\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

with $\alpha, \beta \in \mathbb{R}$ and $b = a - 1 \neq 0$.

Algebra T3

- $Ann(A) = \{0\}$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{f_1, f_2\}$, $\mathcal{I}(A) = \emptyset$,
- $A^2 = A$,

- $Der A = \mathbb{R}D_1 \oplus \mathbb{R}D_2$, where $[D_1]_{B'} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $[D_2]_{B'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- $[D_1, D_2] = 0$ and $\tilde{D} = \frac{\theta}{\gamma}D_1 - \frac{\beta\theta}{\alpha\gamma}D_2$,
- $Aut A = \{T(\alpha, \beta) \mid \forall \alpha \in \mathbb{R}\}$ where

$$T(\alpha, \beta) = \begin{bmatrix} 1 & 2\alpha & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix}$$

with $\alpha, \beta \in \mathbb{R}$.

Algebra T4

- $Ann(A) = \{0\}$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{f_1, f_2\}$, $\mathcal{I}(A) = \emptyset$,
- $A^2 = A$,

- $Der A = \mathbb{R}D_1 \oplus \mathbb{R}D_2 \oplus \mathbb{R}D_3 \oplus \mathbb{R}D_4$, where

$$[D_1]_{B'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [D_2]_{B'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[D_3]_{B'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [D_4]_{B'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

- $[D_1, D_2] = 0$, $[D_1, D_3] = D_1$, $[D_1, D_4] = D_2$, $[D_2, D_3] = 0$,
 $[D_2, D_4] = 0$, $[D_3, D_4] = D_4$ and $\tilde{D} = \frac{\gamma}{\alpha}D_1 + \frac{\theta}{\gamma}D_4$,
- $Aut A = \{T(\alpha, \beta, \gamma, \delta) \mid \forall \alpha \in \mathbb{R}\}$ where

$$T(\alpha, \beta, \gamma, \delta) = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & \gamma & \delta \\ 0 & 0 & 1 \end{bmatrix}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Algebra T5

- $Ann(A) = \{0\}$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{f_1, f_2\}$,

$$\mathcal{I}(A) = \left\{ xf_1 + yf_2 + \frac{1}{2}f_3 \mid \forall x, y \in \mathbb{R} \right\},$$

- $A^2 = A$,
- $Der A = \mathbb{R}D_1 \oplus \mathbb{R}D_2 \oplus \mathbb{R}D_3 \oplus \mathbb{R}D_4 \oplus \mathbb{R}D_5 \oplus \mathbb{R}D_6$, where

$$[D_1]_{B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [D_2]_{B'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [D_3]_{B'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[D_4]_{B'} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [D_5]_{B'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [D_6]_{B'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

- $[D_1, D_2] = D_2$, $[D_1, D_3] = D_3$, $[D_1, D_4] = -D_4$, $[D_1, D_5] = 0$,
 $[D_1, D_6] = 0$, $[D_2, D_3] = 0$, $[D_2, D_4] = D_1 - D_5$, $[D_2, D_5] = D_2$,
 $[D_2, D_6] = D_3$, $[D_3, D_4] = -D_3$, $[D_3, D_5] = 0$, $[D_3, D_6] = 0$,
 $[D_4, D_5] = -D_4$, $[D_4, D_6] = 0$, $[D_5, D_6] = D_6$,
and $\tilde{D} = \frac{\gamma}{\alpha}D_2 - \frac{\beta\theta}{\alpha\gamma}D_3 + \frac{\theta}{\gamma}D_4$,
- $Aut A = \{T(\alpha, \beta, \gamma, \delta, \varepsilon, \theta) \mid \forall \alpha \in \mathbb{R}\}$ where

$$T(\alpha, \beta, \gamma, \delta, \varepsilon, \theta) = \begin{bmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & 0 & 1 \end{bmatrix}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha\varepsilon - \beta\delta \neq 0$.

Algebra T6

- $Ann(A) = \mathbb{R}f_1$, $\mathcal{N}(A) = \mathbb{R}f_1$, $\mathcal{I}(A) = \emptyset$,
- $A^2 = A$,
- $Der A = \mathbb{R}\tilde{D}$,
- $Aut A = \{T(\alpha) \mid \forall \alpha \in \mathbb{R}\}$ where

$$T(\alpha) = \begin{bmatrix} 1 & \alpha & \frac{1}{2}(\alpha^2 - \alpha) \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix}.$$

Algebra T7

- $Ann(A) = \mathbb{R}f_1$, $\mathcal{N}(A) = \mathbb{R}f_1$, $\mathcal{I}(A) = \left\{ y^2f_1 + yf_2 + \frac{1}{2}f_3 \mid \forall y \in \mathbb{R} \right\}$,
- $A^2 = A$,

- $Der A = \mathbb{R}D_1 \oplus \mathbb{R}D_2$ where $[D_1]_{B'} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $[D_2]_{B'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- $[D_1, D_2] = D_2$ and $\tilde{D} = \frac{\gamma}{\alpha}D_2$,
- $Aut A = \{T(\alpha, \beta) \mid \forall \alpha \in \mathbb{R}\}$ where

$$T(\alpha, \beta) = \begin{bmatrix} \alpha^2 & \alpha\beta & \frac{1}{2}\beta^2 \\ 0 & \alpha & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$.

Algebra T8

- $Ann(A) = \mathbb{R}f_1$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{f_1, f_2\}$, $\mathcal{I}(A) = \emptyset$,
- $A^2 = A$,
- $Der A = \mathbb{R}D_1 \oplus \mathbb{R}D_2 \oplus \mathbb{R}D_3$ where

$$[D_1]_{B'} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [D_2]_{B'} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } [D_3]_{B'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $[D_1, D_2] = D_2$, $[D_1, D_3] = 2D_3$, $[D_2, D_3] = 0$ and $\tilde{D} = \frac{\gamma}{\alpha}D_2$,
- $Aut A = \{T(\alpha, \beta) \mid \forall \alpha \in \mathbb{R}\}$ where

$$T(\alpha, \beta, \gamma) = \begin{bmatrix} \alpha^3 & 2\alpha\beta & \gamma \\ 0 & \alpha^2 & \beta \\ 0 & 0 & \alpha \end{bmatrix}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha \neq 0$.

Algebra T9

- $Ann(A) = Span_{\mathbb{R}}\{f_1, f_2\}$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{f_1, f_2\}$, $\mathcal{I}(A) = \emptyset$,
- $A^2 = \mathbb{R} \ker \tilde{D}$,
- $Der A = \mathbb{R}D_1 \oplus \mathbb{R}D_2 \oplus \mathbb{R}D_3 \oplus \mathbb{R}D_4 \oplus \mathbb{R}D_5$ where

$$[D_1]_{B'} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [D_2]_{B'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [D_3]_{B'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[D_4]_{B'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, [D_5]_{B'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

- $[D_1, D_2] = 0$, $[D_1, D_3] = 2D_3$, $[D_1, D_4] = -D_4$, $[D_1, D_5] = D_5$,
 $[D_2, D_3] = -D_3$, $[D_2, D_4] = D_4$, $[D_2, D_5] = 0$, $[D_3, D_4] = D_5$,

$$[D_3, D_5] = 0, [D_4, D_5] = 0 \text{ and } \tilde{D} = \frac{\gamma}{\alpha}(2D_3 + D_4)$$

- *Aut* $A = \{T(\alpha, \beta) \mid \forall \alpha \in \mathbb{R}\}$ where

$$T(\alpha, \beta, \gamma, \delta, \varepsilon) = \begin{bmatrix} \alpha^2 & \gamma & \varepsilon \\ 0 & \beta & \delta \\ 0 & 0 & \alpha \end{bmatrix}$$

with $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{R}$, $\alpha\beta \neq 0$.

The Isomorphism Problem. Before dealing with the algebras $T_1 - T_9$ it must to assure that do not exist among them pairs of isomorphic algebras. A first evaluation can be made taking in account the fact that an isomorphism carries any special element of the algebra (annulator, nilpotent, idempotent) on a special element of the same kind from the image. Recall also that two isomorphic algebras have isomorphic automorphism groups as well as isomorphic LIE derivation algebras.

Using now these remarks and the before listed properties of algebras it can be proved the following result.

Proposition 3.4 *Any algebra T_i is not isomorphic with any algebra T_j , where $i, j \in \{1, 2, \dots, 9\}$ and $i \neq j$.*

4 Classification of quadratic differential systems on \mathbb{R}^3 having a nilpotent of order 3 derivative

By the usual procedure we associate with the algebra $A(T_i)$ the HQDS S_i , $i = 1, 2, \dots, 9$. We list below these systems together with their main properties.

$$\text{System } S_1 \quad \begin{cases} \dot{x}^1 = 2x^1x^3 + (x^2)^2 \\ \dot{x}^2 = 2ax^2x^3 + (x^3)^2 \\ \dot{x}^3 = 2a(x^3)^2 \end{cases}$$

- The vector field associated with it is

$$Y = (2x^1x^3 + (x^2)^2)\frac{\partial}{\partial x^1} + (2ax^2x^3 + (x^3)^2)\frac{\partial}{\partial x^2} + 2a(x^3)^2\frac{\partial}{\partial x^3}.$$

- The infinitesimal generator of its automorphism group is

$$X = -(x^2 - \frac{1}{2b}x^3)\frac{\partial}{\partial x^1} - bx^3\frac{\partial}{\partial x^2}.$$

- $[X, Y] = 0$, i.e. X is an element of the centralizer of Y in the LIE algebra of all polynomial vector fields in 3-dimensional space.

$$\text{System } S_2 \quad \begin{cases} \dot{x}^1 = 2x^1x^3 + (x^2)^2 \\ \dot{x}^2 = 2ax^2x^3 \\ \dot{x}^3 = 2a(x^3)^2 \end{cases}$$

- The vector field associated with it is

$$Y = (2x^1x^3 + (x^2)^2)\frac{\partial}{\partial x^1} + 2ax^2x^3\frac{\partial}{\partial x^2} + 2a(x^3)^2\frac{\partial}{\partial x^3}.$$

- The infinitesimal generators of its automorphism group are

$$\begin{aligned} X_1 &= -\frac{1}{b}x^2\frac{\partial}{\partial x^1} - x^3\frac{\partial}{\partial x^2}, \\ X_2 &= -2x^1\frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^2}. \end{aligned}$$

• $[X_1, X_2] = -X_1$, i.e. the LIE algebra generated by X_1, X_2 is isomorphic with the LIE algebra $Der(A(T^2))$.

• $[X_1, Y] = 0$, $[X_2, Y] = 0$, i.e. X_1, X_2 are elements of the centralizer of Y in the LIE algebra of all polynomial vector fields in 3-dimensional space.

$$\text{System } \mathbf{S}_3 \quad \begin{cases} \dot{x}^1 = 2x^1x^3 + 2x^2x^3 \\ \dot{x}^2 = 2x^2x^3 + (x^3)^2 \\ \dot{x}^3 = 2(x^3)^2 \end{cases}$$

- Its associated vector field is

$$Y = 2(x^1x^3 + x^2x^3)\frac{\partial}{\partial x^1} + (2x^2x^3 + (x^3)^2)\frac{\partial}{\partial x^2} + 2(x^3)^2\frac{\partial}{\partial x^3}.$$

- The infinitesimal generators of its automorphism group are

$$\begin{aligned} X_1 &= -2x^2\frac{\partial}{\partial x^1} - x^3\frac{\partial}{\partial x^2}, \\ X_2 &= -x^3\frac{\partial}{\partial x^1}. \end{aligned}$$

• $[X_1, X_2] = 0$, i.e. the LIE algebra spanned by X_1, X_2 is ABELIAN.

• $[X_1, Y] = 0$, $[X_2, Y] = 0$, i.e., X_1 and X_2 belong to the centralizer of Y in the LIE algebra of all polynomial vector fields in space.

$$\text{System } \mathbf{S}_4 \quad \begin{cases} \dot{x}^1 = 2x^1x^3 + (x^3)^2 \\ \dot{x}^2 = 2x^2x^3 \\ \dot{x}^3 = 2(x^3)^2 \end{cases}$$

- Its associated vector field is

$$Y = (2x^1x^3 + (x^3)^2)\frac{\partial}{\partial x^1} + 2x^2x^3\frac{\partial}{\partial x^2} + 2(x^3)^2\frac{\partial}{\partial x^3}.$$

- The infinitesimal generators of its automorphism group are

$$\begin{aligned} X_1 &= -x^2\frac{\partial}{\partial x^1}, & X_2 &= -x^3\frac{\partial}{\partial x^1} \\ X_3 &= -x^2\frac{\partial}{\partial x^2}, & X_4 &= -x^3\frac{\partial}{\partial x^2}. \end{aligned}$$

- The LIE algebra generated by X_1, \dots, X_4 has the multiplication table:

$$\begin{aligned} [X_1, X_2] &= 0, & [X_1, X_3] &= X_1, & [X_1, X_4] &= X_2, \\ [X_2, X_3] &= 0, & [X_2, X_4] &= 0, & [X_3, X_4] &= X_4. \end{aligned}$$

It is isomorphic with the LIE algebra $Der(A(T4))$.

- $[X_1, Y] = 0, [X_2, Y] = 0, [X_3, Y] = 0, [X_4, Y] = 0$, i.e. X_1, X_2, X_3, X_4 belong to the centralizer of Y in the LIE algebra of all polynomial vector fields in space.

System S_5

$$\begin{cases} \dot{x}^1 = 2x^1x^3 \\ \dot{x}^2 = 2x^2x^3 \\ \dot{x}^3 = 2(x^3)^2 \end{cases}$$

- Its associated vector field is

$$Y = 2x^1x^3 \frac{\partial}{\partial x^1} + 2x^2x^3 \frac{\partial}{\partial x^2} + 2(x^3)^2 \frac{\partial}{\partial x^3}.$$

- The associated infinitesimal generators of its automorphism group are

$$\begin{aligned} X_1 &= -x^1 \frac{\partial}{\partial x^1}, & X_2 &= -x^2 \frac{\partial}{\partial x^1}, & X_3 &= -x^3 \frac{\partial}{\partial x^1}, \\ X_4 &= -x^1 \frac{\partial}{\partial x^2}, & X_5 &= -x^2 \frac{\partial}{\partial x^2}, & X_6 &= -x^3 \frac{\partial}{\partial x^2}. \end{aligned}$$

- The LIE algebra generated by X_1, \dots, X_6 has the following multiplication table:

$$\begin{aligned} [X_1, X_2] &= X_2 & [X_1, X_3] &= X_3 & [X_1, X_4] &= -X_4 \\ [X_1, X_5] &= 0 & [X_1, X_6] &= 0 & [X_2, X_3] &= 0 \\ [X_2, X_4] &= X_1 - X_5 & [X_2, X_5] &= X_2 & [X_2, X_6] &= X_3 \\ [X_3, X_4] &= -X_3 & [X_3, X_5] &= 0 & [X_3, X_6] &= 0 \\ [X_4, X_5] &= -X_4 & [X_4, X_6] &= 0 & [X_5, X_6] &= X_6. \end{aligned}$$

It is isomorphic with the LIE algebra $Der(A(T5))$.

- $[X_1, Y] = 0, [X_2, Y] = 0, [X_3, Y] = 0, [X_4, Y] = 0, [X_5, Y] = 0, [X_6, Y] = 0$, i.e., X_1, X_2, \dots, X_6 belong to the centralizer of Y in the LIE algebra of all polynomial vector fields in space.

System S_6

$$\begin{cases} \dot{x}^1 = (x^2)^2 \\ \dot{x}^2 = 2x^2x^3 + (x^3)^2 \\ \dot{x}^3 = 2(x^3)^2 \end{cases}$$

- Its associated vector field is

$$Y = (x^2)^2 \frac{\partial}{\partial x^1} + [2x^2x^3 + (x^3)^2] \frac{\partial}{\partial x^2} + 2(x^3)^2 \frac{\partial}{\partial x^3}.$$

- The associated infinitesimal generator of its uniparametric automorphism group is

$$X = - \left(x^2 - \frac{1}{2} x^3 \right) \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial x^2}.$$

- $[X, Y] = 0$, i.e., X_1 belongs to the centralizer of Y in the LIE algebra of all polynomial vector fields in space.

System S_7

$$\begin{cases} \dot{x}^1 = (x^2)^2 \\ \dot{x}^2 = 2x^2 x^3 \\ \dot{x}^3 = 2(x^3)^2 \end{cases}$$

- Its associated vector field is

$$Y = (x^2)^2 \frac{\partial}{\partial x^1} + 2x^2 x^3 \frac{\partial}{\partial x^2} + 2(x^3)^2 \frac{\partial}{\partial x^3}.$$

- The associated infinitesimal generators of its automorphism group are

$$\begin{aligned} X_1 &= -2x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2}, \\ X_2 &= -x^2 \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial x^2}. \end{aligned}$$

- $[X_1, X_2] = X_2$, i.e. the LIE algebra generated by X_1, X_2 is isomorphic with the LIE algebra $Der(A(T^7))$.

- $[X_1, Y] = 0$, $[X_2, Y] = 0$, i.e., X_1, X_2 belong to the centralizer of Y in the LIE algebra of all polynomial vector fields in space.

System S_8

$$\begin{cases} \dot{x}^1 = 2x^2 x^3 \\ \dot{x}^2 = (x^3)^2 \\ \dot{x}^3 = 0 \end{cases}$$

- Its associated vector field is

$$Y = 2x^2 x^3 \frac{\partial}{\partial x^1} + (x^3)^2 \frac{\partial}{\partial x^2}.$$

- The associated infinitesimal generators of its automorphism group are

$$\begin{aligned} X_1 &= -3x^1 \frac{\partial}{\partial x^1} - 2x^2 \frac{\partial}{\partial x^2} - x^3 \frac{\partial}{\partial x^3}, \\ X_2 &= -2x^2 \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial x^2}, \\ X_3 &= -x^3 \frac{\partial}{\partial x^1}. \end{aligned}$$

- The LIE algebra generated by X_1, X_2, X_3 has the following multiplication table:

$$[X_1, X_2] = X_2, \quad [X_1, X_3] = 2X_3, \quad [X_2, X_3] = 0.$$

It is isomorphic with the LIE algebra $Der(A(T8))$.

- $[X_1, Y] = 0, [X_2, Y] = 0, [X_3, Y] = 0$, i.e., X_1, X_2, X_3 belong to the centralizer of Y in the LIE algebra of all polynomial vector fields in space.

System S_9

$$\begin{cases} \dot{x}^1 = (x^3)^2 \\ \dot{x}^2 = 0 \\ \dot{x}^3 = 0 \end{cases}$$

- Its associated vector field is

$$Y = (x^3)^2 \frac{\partial}{\partial x^1}.$$

- The associated infinitesimal generators of its automorphism group are

$$\begin{aligned} X_1 &= -2x^1 \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial x^3}, \\ X_2 &= -x^2 \frac{\partial}{\partial x^2}, & X_3 &= -x^2 \frac{\partial}{\partial x^1}, \\ X_4 &= -x^3 \frac{\partial}{\partial x^1}, & X_5 &= -x^3 \frac{\partial}{\partial x^2}. \end{aligned}$$

- The LIE algebra generated by X_1, X_2, X_3, X_4, X_5 has the following multiplication table:

$$\begin{aligned} [X_1, X_2] &= 0, & [X_1, X_3] &= 2X_3, & [X_1, X_4] &= -X_4, \\ [X_1, X_5] &= X_5, & [X_2, X_3] &= -X_3, & [X_2, X_4] &= X_2, \\ [X_2, X_5] &= 0, & [X_3, X_4] &= X_5, & [X_3, X_5] &= 0, \\ [X_4, X_5] &= 0. \end{aligned}$$

It is isomorphic with the LIE algebra $Der(A(T8))$.

- $[X_1, Y] = 0, [X_2, Y] = 0, [X_3, Y] = 0, [X_4, Y] = 0, [X_5, Y] = 0$, i.e., X_1, X_2, X_3, X_4, X_5 belong to the centralizer of Y in the LIE algebra of all polynomial vector fields in space.

5 Concluding Remarks

1. There exist nine classes of affinely nonequivalent HQDSs having a nilpotent of order 3 derivation.
2. In accordance with the general theory (see [5]), the derivation algebras and the automorphism groups of these systems are coincident with the similar objects of the associated commutative algebras. Moreover, it was pointed out that these derivation algebras are just the LIE algebras of the connex component of the identity of the corresponding automorphisms groups, what agrees with the general theory (see [7]).

3. The general solution of each such a system can be easily obtained. Indeed, the general solution $x^3(t)$ of the third equation can be easily obtained. Putting $x^3(t)$ instead of x^3 in the second equation, a linear equation and its general solution $x^2(t)$ are obtained. Finally, the first equation becomes also a linear equation after changing x^2 and x^3 in $x^2(t)$ and $x^3(t)$, respectively.

4. All derivatives of algebras T1-T9 have only real eigenvalues.

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