POSITIVE SOLUTIONS TO DIRICHLET PROBLEMS WITH SINGULAR AND SUPERLINEAR TERMS

Sergiu Aizicovici, Nikolaos S. Papageorgiou, and **Vasile Staicu**
Department of Mathematics, Ohio University, Athens, OH 45701, USA
Department of Mathematics, National Technical University, Athens 15780, Greece
Department of Mathematics and CIDMA, University of Aveiro, 3810-193 Aveiro, Portugal
vasile@ua.pt

Abstract: Weconsider a parametricnonlinearDirichlet problem driven by the p–Laplacian, with a singular term and a p–superlinear perturbation, which needs not satisfy the usual in such cases Ambrosetti-Rabinowitz condition. Using variational methods together with truncation techniques, we prove a bifurcation-type theorem describing the behavior of the set of positive solutions as the parameter varies.

1.Introduction

Let Ω be a boundeddomainwith a C^2–boundary $\partial\Omega$.
We consider the followingnonlinear Dirichlet problem

$$(P_\lambda)\begin{cases}-\Delta_p u(z) = \lambda\xi(z)u(z)^{-\eta} + f(z,u(z)) in\Omega \\ u(z) = 0 \; on\, \partial\Omega,\end{cases}$$

where$\lambda > 0$ is a parameter, $\eta\in (0,1)$, Δ_p denotes the p-Laplace differential operator defined by

$\Delta_p u = div(\|Du\|^{p-2}Du, for\ u \in W_0^{1,p}(\Omega), 1 < p < \infty,$

and$f(z.x)$ is a Carathéodory perturbation such that $x \to f(z,x)$ is $(p-1)$ – superlinearnear$+\infty$.

So, in (P_λ) we have a combined effect of a singular term and of a superlinear perturbation. Ourgoalis to study the dependence of the set of positive solutions on the parameter$\lambda > 0$.

In the past, parametricproblemsexhibiting the competingeffects of differentkinds of nonlinearitieswerefirststudiedbyAmbrosetti-Brezis-Cerami[5], who proved bifurcation type results for semilinear problems (i.e., p = 2) with a reaction of the form

$$f(z,x) = \lambda x^{q-1} + x^{r-1}, x>0, 1<q<2<r<2^*,$$

wherewe have the combined effect of a concave termλx^{q-1} and of the convex term x^{r-1}.

Their work was extended to nonlinear equations by Garcia Azorero-Manfredi-PeralAlonso[8], Guo-Zhang [12] and Hu-Papageorgiou[15].

Other papers studying the combined effect of singular and superlinear terms are those ofCoclite-Palmieri [8], Ghergu-Radulescu [9], Haitao [13], Hirano-Saccon-Shioji [14], Sun-Zhang [17], Sun-Wu-Lang [18]and of Giacomoni-Schindler-Takac [10] and Perera-Zhang [16].

However, in the aforementioned papers, either the perturbation has a special form or the multiplicity theoremproved does not give the precise dependence of the set of positive solutions on theparameter$\lambda > 0$.

Finally, we mention that the problem(P_λ) with Neumann boundary condition has been studied in the recent paper [4], where by using differentmethods, we obtained a multiplicity result, but not a bifurcation type theorem.

The approach in this paper is variational, based on the critical point theory and also usestruncations and comparison techniques.

2. Mathematical background. Let$(X, \|.\|)$ be a Banach space, X^* beits dual and let $\langle.,.\rangle$be the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$.

We saythat φ satisfies the Cerami condition if for every sequence $(x_n)_{n\geq 1}$ in X such that $(\varphi(x_n))_{n\geq 1}$ is bounded in R and $(1 + \|x_n\|)\varphi'(x_n) \to 0\ in X^* as n \to \infty$,admits a strongly convergent subsequence.

This compactness-type condition suffices to prove a deformation theorem and from it derive the minimax theory of certain critical values of φ,in particular the well-known mountain pass theorem:

Theorem 1. If $(X, \|.\|)$ is a Banach space, $\varphi \in C^1(X)$. satisfies the Cerami condition, $x_0, x_1 \in X$ and $\rho > 0$ are such that $\|x_1 - x_0\| > \rho$, $max\{\varphi(x_0), \varphi(x_1)\} < inf\{\varphi(x): \|x - x_0\| = \rho\} =: \eta_\rho$ and

c:=$inf_{\gamma\in\Gamma} max_{t\in[0,1]}\varphi(\gamma(t))$ where

$\Gamma = \{\gamma \in C([0,1], X): \gamma(0) = x_0, \gamma(1) = x_1\}$

Then $c \geq \eta_\rho$ and c is a criticalvalue of φ.

We know (see, for example [13]) that the negative Dirichlet p-Laplacian has a smallest eigenvalue $\hat{\lambda}_1$>0 which is simple and isolated. Let $\hat{u}_1$ be theL^p-normalized eigenfunction corresponding to $\hat{\lambda}_1$. Then $\hat{u}_1$ has constant sign and in fact $\hat{u}_1 \in intC_+$,where $intC_+$ denotes the interior of the positive cone $C_+$of the ordered Banach space $C_0^1(\Omega) := \{u \in C^1(\overline{\Omega}): u(x) = 0\ on\, \partial\Omega\}$.

3. Positive solutions

The hypotheses on the data of our problem (P_λ) are the following:

$\mathbf{H(\xi)}: \xi \in L^\infty(\Omega), essinf_\Omega\, \xi > 0\ and\ 0 < \eta < \min\{1, \frac{p}{N}\}.$

$\mathbf{H(f)}$: $f: \Omega x R \to R$is a Carathéodory function such that for almost all $z\in\Omega$, $f(z,0) = 0, f(z,x) \geq 0$ for all $x > 0$ and:

(i)
$f(z,x) \leq a(z) + cx^{r-1}$ *for almost all* $z \in \Omega$, all $x \geq 0$, with $a \in L^{\infty}(\Omega), c > 0. p < r < p^{*}$
;
(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for *almost all* $z \in \Omega$;
(iii) there exists $\mu \in ((r-p)\max\left\{1, \frac{N}{p}\right\}, p^{*})$ and $\beta_0>0$ such that $\beta_0 \leq \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^{\mu}} = +\infty$
uniformly for *almost all* $z \in \Omega$;
(iv) there exists $\theta \in L^{\infty}(\Omega), \theta(z) \leq \hat{\lambda}_1$ *a.e. in* Ω, $\theta(z) \neq \hat{\lambda}_1$ and $\limsup_{x\to 0} \frac{f(z,x)}{x^{p-1}} \leq \theta(z)$ uniformly for *almost all* $z \in \Omega$;
(v) for every $\rho > 0$ there exists $\delta_{\rho} > 0$ such that for almost all $z \in \Omega$, the map $x \to f(z,x) + \delta_{\rho} x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark: Assumption **H(f)**(ii) implies that for almost all $z \in \Omega$ the primitive $x \to F(z,x)$ is p-supelinear near $+\infty$. However, note that we do not employ the usual in such cases AR-condition.

Example: The function
$$f(z,x) = x^{p-1}\left(\ln(1+x) + \frac{x}{p(1+x)}\right) \text{for } x \geq 0$$
satisfies hypotheses H(f) but not the AR-condition.

Our main result is the following bifurcation theorem:

Theorem 2 If hypotheses **H(ξ)** and **H(f)** hold, then there exists $\lambda^{*}\in(0,+\infty)$ such that:
(a) for every $\lambda\in(0,\lambda^{*})$, problem (P_{λ}) has at least two positive solutions
$$u_0, \hat{u}\in \text{int } C_+, u_0\leq \hat{u}, u_0\neq \hat{u};$$
(b) for $\lambda=\lambda^{*}$, problem (P_{λ}) has at least one positive solution $\hat{u}^{*}\in \text{int } C_+$;
(c) for $\lambda>\lambda^{*}$, problem (P_{λ}) has no positive solution.

The proof of Theorem 2 relies on a series of propositions of independent interest. First we consider the auxiliary problem
$$(Q_{\lambda})\begin{cases} -\Delta_p u(z) = \lambda\xi(z)u(z)^{-\eta} in \Omega \\ u(z) = 0 \ on \ \partial\Omega, \end{cases}$$

Proposition 3 If hypotheses **H(ξ)** and **H(f)** hold and $\lambda > 0$ then problem (Q_{λ}) has a unique solution $\underline{u} \in intC_+$.

The proof of Proposition 3 is based on the method of "upper-lower" solutions.

Hypotheses **H(f)** imply that, given $\varepsilon > 0$ we can find $\bar{C}=\bar{C}(\varepsilon) > 0$ such that
$0 \leq f(z,x) \leq (\theta(z) + \varepsilon)x^{p-1} + \bar{C}x^{r-1}$ *for a.a.* $z \in \Omega$, all $x \geq 0$.

So we are led to consider the following auxiliary Dirichlet problem
$$(S_{\lambda}^{\varepsilon})\begin{cases} -\Delta_p u = \lambda\xi\underline{u}(z)^{-\eta} + (\theta+\varepsilon)u^{p-1} + \bar{C}u^{r-1} in \Omega \\ u(z) = 0 \ on \ \partial\Omega, \end{cases}$$

Using variational methods we show that for suitably small $\varepsilon, \lambda > 0$ problem $(S_{\lambda}^{\varepsilon})$ has a solution. To this end we introduce the Carathéodory function
$$\beta_{\lambda}^{\varepsilon}(z,x) = \begin{cases} \lambda\xi\underline{u}(z)^{-\eta} + (\theta+\varepsilon)x^{p-1} + \bar{C}x^{r-1} & if \ x > 0 \\ 0 & if \ x \leq 0. \end{cases}$$

Set $B_{\lambda}^{\varepsilon}(z,x) = \int_0^x \beta_{\lambda}^{\varepsilon}(z,s)\,ds$ and consider the energy functional of problem $(S_{\lambda}^{\varepsilon})$ defined by
$$\psi_{\lambda}^{\varepsilon}(u) = \frac{1}{p}\|Du\|_p^p - \int_{\Omega} B_{\lambda}^{\varepsilon}(z,u(z))dz \text{ for } u \in W_0^{1,p}(\Omega).$$

Proposition 4 If hypotheses **H(ξ)** and **H(f)** hold and $\lambda > 0$ then we can find $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0,\hat{\varepsilon})$, *the functional* $\psi_{\lambda}^{\varepsilon}$ satisfies the Cerami condition.

The next proposition verfies the mountain pass geometry for the functional $\psi_{\lambda}^{\varepsilon}$.

Proposition 5 If hypotheses **H(ξ)** and **H(f)** hold and $\varepsilon \in (0,\hat{\varepsilon})$, then we can find $\hat{\lambda}>0$ and $\rho>0$ small such that
$\psi_{\lambda}^{\varepsilon}(u) \geq \gamma_0>0$ for all $u\in W_0^{1,p}(\Omega)$.

Since $r > p$ we have:

Proposition 6 If hypotheses **H(ξ)** and **H(f)** hold and $\varepsilon, \lambda > 0$ then $\psi_{\lambda}^{\varepsilon}(t\hat{u}) \to -\infty$ *as* $t \to +\infty$.

Propositions 4, 5, 6 and Theorem 1 (the Mountain Pass Theorem) lead to:

Proposition 7 If hypotheses **H(ξ)** and **H(f)** hold, $\varepsilon \in (0,\hat{\varepsilon})$ and $\lambda \in (0,\hat{\lambda})$ then problem $(S_{\lambda}^{\varepsilon})$ has a positive solution $\bar{u} \in intC_+$ with $\underline{u} \leq \bar{u}$.

Let

$$\pounds = \{\lambda > 0 : (P_{\lambda}) \text{ has a nontrivial positive solution}\}.$$

Using truncation techniques and the method of upper-lower solutions we have:

Proposition 8 If hypotheses **H(ξ)** and **H(f)** hold then $\pounds \neq \emptyset$.

Let
$\lambda^* = \sup \pounds$.

Proposition 9 If hypotheses **H(ξ)** and **H(f)** hold then

$\lambda^* < \infty$.

The next proposition can be viewed as a "singular" version of analogous results obtained earlier by Guedda-Veron [11] and Arcoya-Ruiz [6]. Also, it extends Theorem 2.3 of Giacomoni-Schindler-Takac [10]. To state the result we need to introduce the following notation.
For $h_1, h_2 \in L^q(\Omega)$, q>1, we write $h_1 \prec h_2$ if, for any compact set $K \subset \Omega$, we can find $\varepsilon=\varepsilon(K)>0$ such that

$h_1(z) + \varepsilon \le h_2(z)$ for a. a. $z \in K$.

Obviously, if $h_1, h_2 \in C(\Omega)$ and $h_1(z)<h_2(z)$ for all $z \in \Omega$, then $h_1 \prec h_2$.

Proposition 10 If $\tau \ge 0$, $h_1, h_2 \in L^q(\Omega)$, (q>1) with $h_1 \prec h_2$, $u \in C_+$ with $u(z)>0$ for all $z \in \Omega$, $v \in$ int C_+ and

$$-\Delta_p u(z) - \lambda\xi(z)u(z)^{-\eta} + \tau u(z)^{p-1} = h_1(z) \text{ in } \Omega,$$
$$-\Delta_p v(z) - \lambda\xi(z)v(z)^{-\eta} + \tau v(z)^{p-1} = h_2(z) \text{ in } \Omega,$$
Then $v-u \in int C_+$.

Proposition 11 If If hypotheses **H(ξ)** and **H(f)** hold and $\lambda \in (0,\lambda^*)$, then problem (P_λ) has at least two positive solutions

u_0, $\hat{u} \in$ int C_+, $u_0 \le \hat{u}$, $u_0 \ne \hat{u}$.

(One solution is obtained via the direct method and the other via the mountain pass theorem).

Proposition 12 If If hypotheses **H(ξ)** and **H(f)** hold and $\lambda=\lambda^*$, then problem (P_λ) has at least one positive solution $\hat{u}^* \in$ int C_+.

Proof of Theorem 2.
The conclusions of Theorem 2 are direct consequences of Propositions 9, 11, 12 and the definition of λ^*.

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