ONE PROBLEM OF RESOURCE ADMINISTRATION

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Abstract: The present work has the aim to develop a handy algorithm for computing the results of a production process with two essential resources, according to Leontief's model. We estimate the production level V_T for a given period of time [0,T] and analyze two possible cases: (1) a part of the entering resources is not used completely during certain intervals of time in the production period, and cannot be used afterwards in the given production process; (2) the unused quantities of the resources are accumulated and can be used further in that production. The study focuses on investigating the second case. The recursion formulas are derived for calculating the production level.

Keywords: Leontief model, production control, production optimization

1. Introduction

Consider a production process that takes place during the period [0,T] and is based on two main resources *X* and *Y*, for example, raw material measured in m^2 and, respectively, labor measured in hours. Suppose that the flows of these resources, available for production, are characterized at each moment $t \in [0,T]$ by their entering rates expressed as the values of the functions $\dot{x}(t)$ and, respectively, $\dot{y}(t)$, which are nonnegative and piecewise continuous in $[0,T]$. Thus, the magnitudes $x(t)=\int_0^t \dot{x}(\tau)d\tau$ and $y(t)=\int_0^t \dot{y}(\tau)d\tau$ represent the quantities of resources *X* and, respectively, *Y* obtained (purchased) to the moment *t*.

Let the expenses of resources *X* and *Y* pet unit of production be represented at each moment $t \in [0,T]$ by the values of the functions $a(t)$ and, respectively, $b(t)$. Suppose that $a(t)$ are positive and piecewise continuous in $[0,T]$. Then the production level obtained to the moment *t* can be calculated using the formula $V(t)=\int_0^t \min\left\{\frac{\dot{x}(\tau)}{a(\tau)},\frac{\dot{y}(\tau)}{b(\tau)}\right\}d\tau$, provided that the surpluses of resources remaining unused in production to the moment *t* will not be implied afterwards in the production process. We will say in such cases that the resources are not accumulated.

2. Estimating the production level

Statement 1. The total production level, V_T, satisfies the following inequality: $V_T = V(T) \le \min\left\{\frac{x(T)}{a},\frac{y(T)}{b}\right\}$, where $a=\min\limits_{t\in[0,T]} a(t)$ and $b=\min\limits_{t\in[0,T]} b(t)$.

Proof. Indeed, to produce the number of V_T units the minimum quantities of resources, $V_T \cdot a$ and, respectively, $V_T \cdot b$ are needed. Obviously, these quantities cannot exceed the amounts of resources available during the whole production period, that is $V_T \cdot a \le x(T)$ and $V_T \cdot b \le y(T)$. These last inequalities imply the relations $V_T \le \frac{x(T)}{a}$ and $V_T \le \frac{y(T)}{b}$, and therefore, $V_T \le \min\left\{\frac{x(T)}{a},\frac{y(T)}{b}\right\}$.

If the resources are not accumulated then at each moment $t \in [0,T]$ the following relation for the production volume $V(t)$ is held:

$$V(t)=\int_0^t \min\left\{\frac{\dot{x}(\tau)}{a(\tau)},\frac{\dot{y}(\tau)}{b(\tau)}\right\}d\tau \le$$

$$\le \min\left\{\int_0^t \frac{\dot{x}(\tau)}{a(\tau)}d\tau,\int_0^t \frac{\dot{y}(\tau)}{b(\tau)}d\tau\right\}.$$

Definition. We will say that the resource *X (Y)* is *deficient* in the period of time $[t_1,t_2]$ if for any $t\in[t_1,t_2]$,

$$\frac{\dot{x}(\tau)}{a(\tau)} < \frac{\dot{y}(\tau)}{b(\tau)} \left(\frac{\dot{x}(\tau)}{a(\tau)} > \frac{\dot{y}(\tau)}{b(\tau)}\right).$$

In each segment of time one of the resources can be deficient. Namely this resource limits the production volume in the mentioned period. Note that a resource can become deficient both due to the magnitude of the resource flow rate $\dot{x}(t)$ or $\dot{y}(t)$ and due to the unit expenses of the resources, $a(t)$ or, respectively, $b(t)$ in the considered period. If during the whole interval $[0,T]$ only one of the resources is deficient, then the production level is calculated according to one of the following formulas.

$$V_T = \int_0^T \frac{\dot{x}(\tau)}{a(\tau)} d\tau \text{ (} X \text{ is deficient),}$$

$$V_T = \int_0^T \frac{\dot{y}(\tau)}{b(\tau)} d\tau \text{ (} Y \text{ is deficient).}$$

Statement 2. If during the production time interval $[0, T]$ the resources X and Y are alternately deficient and the unused resources are not accumulated, then

$$V_T < \min\left\{ \int_0^T \frac{\dot{x}(\tau)}{a(\tau)} d\tau, \int_0^T \frac{\dot{y}(\tau)}{b(\tau)} d\tau \right\}.$$

Proof. Let the resource X be deficient in the interval of time $[0, \Theta)$ and the resource Y be deficient in the interval $[\Theta, T]$, where $0 < \Theta < T$. It means that $\frac{\dot{x}(t)}{a(t)} < \frac{\dot{y}(t)}{b(t)}$ for $t \in [0, \Theta)$ and $\frac{\dot{x}(t)}{a(t)} > \frac{\dot{y}(t)}{b(t)}$ for $t \in [\Theta, T]$.

It follows that during the first period $[0, \Theta)$ the number of $V_1 = \int_0^\Theta \frac{\dot{x}(\tau)}{a(\tau)} d\tau$ units will be produced, while in the second period $[\Theta, T]$ there will be produced $V_2 = \int_\Theta^T \frac{\dot{y}(\tau)}{b(\tau)} d\tau$ units.

Thus, the total production volume is $V_t = V_1 + V_2$, and we have the following inequalities

$$V_T = \int_0^\Theta \frac{\dot{x}(\tau)}{a(\tau)} d\tau + \int_\Theta^T \frac{\dot{y}(\tau)}{b(\tau)} d\tau < \int_0^\Theta \frac{\dot{y}(\tau)}{b(\tau)} d\tau + \int_\Theta^T \frac{\dot{y}(\tau)}{b(\tau)} d\tau = = \int_0^T \frac{\dot{y}(\tau)}{b(\tau)} d\tau$$ and

$$V_T = \int_0^\Theta \frac{\dot{x}(\tau)}{a(\tau)} d\tau + \int_\Theta^T \frac{\dot{y}(\tau)}{b(\tau)} d\tau < \int_0^\Theta \frac{\dot{x}(\tau)}{a(\tau)} d\tau + \int_\Theta^T \frac{\dot{x}(\tau)}{a(\tau)} d\tau = = \int_0^T \frac{\dot{x}(\tau)}{a(\tau)} d\tau.$$

Consequently, $V_T < \min\left\{ \int_0^T \frac{\dot{x}(\tau)}{a(\tau)} d\tau, \int_0^T \frac{\dot{y}(\tau)}{b(\tau)} d\tau \right\}$.

From this the following result.

1. If the resources are not accumulated, the production level can be controlled by means of alternating the resources deficiencies.

2. If the surpluses of resources can be accumulated and used afterwards in the production process, then they can yield an increase in the total production volume. We will illustrate this fact in the following simple example.

Example 1. Consider the production process of two periods length, each period being equal to one day. Let there be given for the resource X:

$$\dot{x}(t) = 10 \ m^2 / day,$$

$$a(t) = 2 \ m^2 / unit \quad for \quad t \in [0,2],$$

and for the resource Y:

$$\dot{y}(t) = \begin{cases} 16 \ hours / day & for \quad t \in [0,1) \\ 24 \ hours / day & for \quad t \in [1,2] \end{cases},$$

$$b(t) = 4 \ hours / unit \quad for \quad t \in [0,2]$$

Calculate the total production level V_T.

Solution. We will analyze two cases.

A. The resources unused in the first day cannot be retained for the second day.

In the first day we have produced

$$V_1 = \min\left\{ \frac{\dot{x}(0)}{a(0)}, \frac{\dot{y}(0)}{b(0)} \right\} \cdot 1 day = = \min\left\{ \frac{10}{2}, \frac{16}{4} \right\} = 4 \ units.$$

In the second day there will be produced

$$V_2 = \min\left\{ \frac{\dot{x}(1)}{a(1)}, \frac{\dot{y}(1)}{b(1)} \right\} \cdot 1 day = = \min\left\{ \frac{10}{2}, \frac{24}{4} \right\} = 5 \ units.$$

Thus, the total production level will be $V_T = V_1 + V_2 = 4 + 5 = 9 \ units$.

B. The resources unused in the first day are accumulated and can be used in the second production day.

Let us calculate how much of each resources was spent in the first day of production:

$\overline{x}(1) = V_1 \cdot a(0) = 4 \cdot 2 = 8 \ m^2$ and

$\overline{y}(1) = V_1 \cdot b(0) = 4 \cdot 4 = 16 \ hours$.

Hence, the quantities of resources remained unused in the first day were

$\Delta x(1) = \dot{x}(0) \cdot 1 day - \overline{x}(1) = 10 - 8 = 2 \ m^2$ and

$\Delta y(1) = \dot{y}(0) \cdot 1 day - \overline{y}(1) = 16 - 16 = 0 \ hours$.

Therefore, the resources available for the second day consist of $\widetilde{x}(1) = \dot{x}(1) \cdot 1 day + \Delta x(1) = 10 + 2 = 12 \ m^2$ and $\widetilde{y}(1) = \dot{y}(1) \cdot 1 day + \Delta y(1) = 24 + 0 = 24 \ hours$.

Consequently, the second day production volume will be $\widetilde{V}_2 = \min\left\{ \frac{\widetilde{x}(1)}{a(1)}, \frac{\widetilde{y}(0)}{b(1)} \right\} = \min\left\{ \frac{12}{2}, \frac{24}{4} \right\} = 6 \ units$.

Thus, the total production level after 2 days will be $V_T = V_1 + \widetilde{V}_2 = 4 + 6 = 10$ *units*.

3. Production level. Computational scheme

Further we generalize the formulas used in Example 1 for the case of *n* consecutive intervals in the production period.

Let δt be the unit of measure of time, say one day. Suppose, that the production period $[0, T]$ consists of an integer number of units δt. Let us denote: $t_0 = 0,\ t_1 = \delta t, \dots t_k = k \cdot \delta t, \dots, t_n = n \cdot \delta t = T$.

Assume, also, that in each interval of time $[t_k, t_{k+1})$ for $k = \overline{0, n-1}$, there are known the values $x_k = x(t_k)$, $y_k = y(t_k)$, $a_k = a(t_k)$ and $b_k = b(t_k)$, that express the amounts of resources and their unit expenses available at the beginning of the respective intervals.

We will denote by $\widetilde{V}_k = \widetilde{V}_k(t_k)$ the production level obtained in the period of time $[t_{k-1}, t_k)$, for $k = \overline{1, n}$, using inclusively the resources accumulated during the preceding interval.

Then, to calculate the volumes $\widetilde{V}_k$ for $k = \overline{1, n}$, the following recursion formulas can be used:

$$\widetilde{V}_{k+1} = \min\left(\frac{\widetilde{x}_k}{a_k}, \frac{\widetilde{y}_k}{b_k}\right), \text{ for } k = \overline{0, n-1} \qquad (1)$$

where
$$\left.\begin{aligned} \widetilde{x}_k &= x_k + \widetilde{x}_{k-1} - \widetilde{V}_k \cdot a_{k-1} \\ \widetilde{y}_k &= y_k + \widetilde{y}_{k-1} - \widetilde{V}_k \cdot b_{k-1} \end{aligned}\right\} \qquad (2)$$

$$\widetilde{x}_0 = x_0, \quad \widetilde{y}_0 = y_0$$

Finally, the total production level in the period $[0, T]$ will be $V_T = \widetilde{V}_1 + \widetilde{V}_2 + \dots + \widetilde{V}_n$.

Note. If during the production process the values x_k, y_k, a_k, and b_k (or part of them) can be controlled, then a possibility of administrating this process appears, with diverse purposes such as: maximization or minimization of the production level during certain periods of time, economizing the resources and so forth.

We will call the values x_k, y_k, a_k, and b_k (given or controlled) for $k = \overline{0, n-1}$ the *production process conditions*, from among which the values x_0, y_0, a_0, and b_0 will be called *process initial conditions*.

In the example below we show the application of the formulas (1)-(2) for a certain production process, and also mention some elements of administrating this process.

Example 2. Consider a production process of 16 days length with the following initial conditions:
$x_0 = 12\,m^2$, $y_0 = 8\,hours$, $a_0 = 3m^2/unit$, $b_0 = 4\,hours/unit$.
Assuming that all unused resources can be accumulated, calculate the production volumes $\widetilde{V}_k$ for $k = \overline{0, 16}$, and the total production level V_T based on the process conditions x_k, y_k, a_k, and b_k (fixed or controlled) given in the Table 1.

Table 1. Production results

k	x_k	y_k	a_k	b_k	$\widetilde{x}_k$	$\widetilde{y}_k$	$\widetilde{V}_{k+1}$	V_T	$\overline{x}_{k+1}$	$\overline{y}_{k+1}$	Δx_{k+1}	Δy_{k+1}
0	12	8	3	4	12	8	2	2	6	8	6	0
1	12	8	3	4	18	8	2	4	6	8	12	0
2	12	8	3	4	24	8	2	6	6	8	18	0
3	12	16	3	4	30	16	4	10	12	16	18	0
4	12	16	3	4	30	16	4	14	12	16	18	0
5	12	16	3	4	30	16	4	18	12	16	18	0
6	12	16	3	2	30	16	8	26	24	16	6	0
7	12	16	3	2	18	16	6	32	18	12	0	4
8	12	16	3	2	12	20	4	36	12	8	0	12
9	20	16	3	2	20	28	6	42	18	12	2	16
10	20	16	3	2	22	32	7	49	21	14	1	18
11	20	16	3	2	21	34	7	56	21	14	0	20
12	20	16	2	3	20	36	10	66	20	30	0	6
13	20	16	2	3	20	22	7	73	14	21	6	1
14	20	16	2	3	26	17	5	78	10	15	16	1
15	0	16	2	3	16	17	5	83	10	15	6	2

Solution. For reasons of facilitating calculation, let us re-write the formulas (1)-(2) as follows.

$$\widetilde{V}_{k+1} = \min(\frac{\widetilde{x}_k}{a_k}, \frac{\widetilde{y}_k}{b_k}), \text{ for } k = \overline{0, n-1},$$

where
$$\left.\begin{aligned} \bar{x}_k &= \widetilde{V}_k \cdot a_{k-1} \\ \bar{y}_k &= \widetilde{V}_k \cdot b_{k-1} \\ \Delta x_k &= \widetilde{x}_{k-1} - \bar{x}_k \\ \Delta y_k &= \widetilde{y}_k - 1 - \bar{y}_k \\ \widetilde{x}_k &= x_k + \Delta x_k \\ \widetilde{y}_k &= y_k + \Delta y_k \end{aligned}\right\} \text{ for } k = \overline{1, n}$$

$\widetilde{x}_0 = x_0$, $\widetilde{y}_0 = y_0$.

The results of calculation according to these formulas are presented in Table 1.

Notes. Analyzing the effect of the conditions x_k, y_k, a_k, and b_k upon the production level and upon the surpluses of the unused resources, we can mention, for example, the following.

1. The increase of the resource Y quantity from the day 4 (k=3) led to increasing the daily production volume and to stabilizing the resource X surplus ($\Delta x_3 = \Delta x_4 = \Delta x_5 = 18$).

2. The diminishing expenses per unit of the Y resource beginning with day 7 (due, for example, to using some labor-saving devices) resulted in an increase of the daily production volume, but at the same time caused an increase of the resource Y surplus.

3. From day 10 (k=9) the quantity of *X* resource has grown to 20 each day, which produced both an increasing of daily production volume and further accumulation of the *Y* resource surplus.

4. Changes in expenses per unit of product beginning with day 13 (due, for example, to rationalizing the using of raw material) led to considerable increase in the production level, but also to the accumulation of the resource *X* surplus.

5. Finally, reducing to zero the *X* resource quantity in day 16 ($x_{15} = 0$) permitted to maximally diminish the surpluses of unused resources to the end of the production process ($\Delta x_{15} = 6$, $\Delta y_{15} = 2$).

The variation in time of the values *x(t), y(t), a(t),* and $b(t)$ may have diverse reasons such as technical, seasonal and of other nature. Knowing these conditions permits prediction of the production process results, while holding the control over the values *x(t), y(t), a(t),* and $b(t)$ offers the possibility of administrating the given process.

References

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